Constructing isostatic frameworks for the ℓ^1 and ℓ^∞ plane

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Abstract

We use a new coloured multi-graph constructive method to prove that if the edge-set of a graph $G = (V, E)$ has a partition into two spanning trees T_1 and T_2 then there is a map $p : V \to \mathbb{R}^2$, $p(v) = (p(v)_1, p(v)_2)$, such that $|p(u)_i - p(v)_i| \geqslant |p(u)_{3-i} - p(v)_{3-i}|$ for every edge uv in T_i ($i = 1, 2$). As a consequence, we solve an open problem on the realisability of minimally rigid bar-joint frameworks in the ℓ^1 or ℓ^∞-plane. We also show how to adapt this technique to incorporate symmetry and indicate several related open problems on rigidity, redundant rigidity and forced symmetric rigidity in normed spaces.

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1 Introduction

A simple graph $G = (V, E)$ with vertices embedded generically in $\mathbb{R}^2$ inherits a natural edge-labelling $\kappa : E \to \{1, 2\}$ whereby an edge (represented by a straight line segment between its embedded vertices) is labelled either 1 or 2 depending on whether the slope m of its affine span satisfies $|m| < 1$ or $|m| > 1$. Simple examples show that not all edge-labellings $\kappa : E \to \{1, 2\}$ are realisable in this way. Motivated by problems in graph rigidity under ℓ^p distance constraints (see [8, 9, 10] for example), we are interested in the realisability of d-tree decompositions in $\mathbb{R}^d$ (see Section 2 for the corresponding notion of realisability when $d > 2$). A *d-tree decomposition* arises from an edge-labelling $\kappa : E \to \{1, 2, \dots, d\}$ when the edge sets $\kappa^{-1}(1), \dots, \kappa^{-1}(d)$ are spanning trees in G. In general, multi-graphs which admit a d-tree decomposition are characterised by the conditions $|E| = d(|V| - 1)$ and $|E(H)| \leqslant d(|V(H)| - 1)$ for each subgraph H (see Nash-Williams [12] and Tutte [16]) and such graphs are said to be *(d, d)-tight*. Constructive characterisations for (d, d)-tight graphs and connections to graph rigidity under ℓ^2 distance constraints are discussed in Tay [14], Frank and Szegö [1] and in Graver, Servatius and Servatius [2, §4.9], for example.

In recent work ([10]) it has been observed that while rigidity properties of graphs under ℓ^1 or ℓ^∞ distance constraints can give rise to special classes of graph decompositions, such as the d-tree decompositions considered here, it is by no means clear as to whether a given graph decomposition always admits a geometric realisation with the specified rigidity property. These realisation problems arise in both symmetric and non-symmetric contexts and are important for several reasons: firstly, they can lead to complete combinatorial characterisations of rigidity; secondly, they provide a method of constructing examples; and thirdly, they allow the existence of geometric frameworks with prescribed rigidity properties to be established by purely combinatorial methods.

In this article we present a constructive method for realising coloured 2-tree decompositions in the plane, thereby solving the (non-symmetric) realisation problem in dimension 2. The method consists of two parts: a multi-graph construction scheme for (d, d)-tight graphs which tracks the evolution of d edge-disjoint spanning trees, and in the case $d = 2$, a method of constructing geometric placements for these multi-graphs which accommodates parallel edges. While it is known that (d, d)-tight graphs are constructible in terms of multi-graphs, the particular role of spanning trees in these constructions is given prominence here. Moreover, the method of assigning geometric placements to multi-graphs used here is a new technique which can be adapted for other contexts.

The graph construction is presented in Section 3 and the geometric realisations for the ℓ^∞-plane are contained in Section 4. In Section 5, we illustrate the versatility of the technique by adapting it to symmetric 2-tree decompositions with no fixed edges, thereby solving the realisation problem in this symmetric context also. The corresponding results for the ℓ^1-plane may be obtained by applying an isometry between the spaces. In the concluding section we state some related open problems and indicate connections to other areas of graph rigidity.

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2 Preliminaries

In this section we introduce terminology, state the main results and provide some background.

2.1 Graph Theory

Let $G = (V, E)$ be a finite, loop-free multi-graph. Let $X \subseteq V$ be a set of vertices. The *neighbourhood* of X in G, $N_G(X)$, is the set of all vertices in $V - X$ which share an edge with some $x \in X$. When $X = \{x\}$ we refer to $N_G(x)$ instead of $N_G(\{x\})$. The subgraph of G *induced* by X is denoted $G[X]$ and has vertex set X and edge set $E_G(X) = \{uv \in E : u, v \in X\}$. We let $i_G(X) = |E_G(X)|$. When the original graph G is apparent from the context, we omit the subscripts and refer simply to $N(X)$, $E(X)$ and $i(X)$. For $F \subseteq E$, the subgraph of G *induced* by F is denoted $G[F]$ and has vertex set $V_G(F) = \{v \in V : uv \in F \text{ for some } u \in V\}$, and edge set F. We say that the edge set F *spans* G when $V_G(F) = V$. Once more, when the graph is apparent, we omit the subscripts. For a vertex $v \in V$, the *degree* of v is the number of edges incident to v, and is denoted by $d_G(v)$ or $d(v)$. Note that for multi-graphs $d_G(v) \geqslant |N_G(v)|$. The *minimum degree of* G is the minimum of $d(v)$ for all $v \in V$ and is denoted $\delta(G)$.

A *d-tree decomposition* is a tuple $\mathcal{G} = (G; T_1, \dots, T_d)$ where G is a multi-graph which is the edge-disjoint union of spanning trees $T_1, \dots, T_d$. Note that since G is a multi-graph, it may have multiple edges between a given pair of vertices $u, v \in V(G)$. As such, we say two spanning trees T_i and T_j of G are *edge-disjoint* if whenever they both contain a uv-edge, these edges are distinct in G. Formally, we regard the tuple $\mathcal{K}_1 = (K_1; T_1, \dots, T_d)$, where K_1 is the graph with a single vertex and no edge, and the edge sets of $T_1, \dots, T_d$ are empty, as a d-tree decomposition. We denote by $\mathcal{G}_d$ the set of all d-tree decompositions. Note that if a multi-graph G admits a d-tree decomposition then it is necessarily loop-free and contains at most d copies of any edge.

2.2 Realisations

Let $G = (V, E)$ be a finite simple graph. A *placement* of G in $\mathbb{R}^d$ is an injective map $p : V \to \mathbb{R}^d$. The pair (G, p) is referred to as a *(bar-joint) framework* in $\mathbb{R}^d$. We let $p(v) = (p(v)_1, \dots, p(v)_d)$ for $v \in V$. Consider a norm on $\mathbb{R}^d$ of the form,

$$\|x\| = \max_{1 \leqslant i \leqslant d} |x \cdot \tilde{e}_i|, \tag{1}$$

where $\tilde{e}_1, \dots, \tilde{e}_d$ is a basis for $\mathbb{R}^d$. For each edge $uv \in E$, the pair $\{p(u), p(v)\}$ is said to be *well-positioned* in $(\mathbb{R}^d, \|\cdot\|)$ if there exists a unique $k \in \{1, \dots, d\}$ such that

$$\|p(u) - p(v)\| = |(p(u) - p(v)) \cdot \tilde{e}_k|.$$

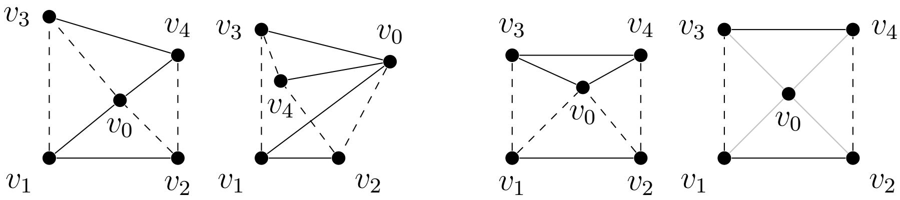


Figure 1: Left: Realisations of two distinct 2-tree decompositions for the wheel graph W_5. Right: A well-positioned placement of W_5 which is not a realisation of a 2-tree decomposition and a placement of W_5 which is not well-positioned.

The framework (G,p) is said to be *well-positioned* if $\{p(u), p(v)\}$ is well-positioned for every edge $uv \in E$.

Each well-positioned framework (G,p) in $(\mathbb{R}^d, \|\cdot\|)$ inherits an edge-labelling $\kappa_p : E \to \{1,\dots,d\}$, referred to as the *framework colouring*, where for each edge $uv \in E$,

$$\kappa_p(uv) = \{k\} \text{ if and only if } \|p(u) - p(v)\| = |(p(u) - p(v)) \cdot \tilde{e}_k|.$$

The set of edges of colour k, $\kappa_p^{-1}(k)$, induces the subgraph $G[\kappa_p^{-1}(k)]$ which is referred to as an induced *monochrome subgraph* of G. A *realisation* in $(\mathbb{R}^d, \|\cdot\|)$ for a d-tree decomposition $\mathcal{G} = (G; T_1, \dots, T_d)$, where G is a simple graph, is a framework (G,p) in $\mathbb{R}^d$ with the property that $T_1, \dots, T_d$ are the induced monochrome subgraphs of G.

To simplify the presentation that follows we will consider only the specific case of the ℓ^∞ norm on $\mathbb{R}^d$. Here we take $(e_i)_{i=1}^d$ to be the usual basis for $\mathbb{R}^d$ and write $\|x\|_\infty = \max_{1\leqslant i\leqslant d} |x \cdot e_i|$ for each $x \in \mathbb{R}^d$. The corresponding statements for other norms of the form (1) are obtained by applying the linear isometry $(\mathbb{R}^d, \|\cdot\|_\infty) \to (\mathbb{R}^d, \|\cdot\|)$, $e_i \mapsto \frac{\tilde{e}_i}{\|\tilde{e}_i\|}$. In particular, the analogous statements for the ℓ^1-plane are obtained by applying the linear isometry $(\mathbb{R}^2, \|\cdot\|_\infty) \to (\mathbb{R}^2, \|\cdot\|_1)$, $(x,y) \mapsto (\frac{x-y}{2}, \frac{x+y}{2})$.

Example 1. Figure 1 illustrates four placements of the wheel graph W_5 in the ℓ^∞-plane together with the induced framework colourings. Note that W_5 admits three distinct 2-tree decompositions (up to graph isomorphism). Two of these are realised by the first two placements pictured on the left of Figure 1. The third placement in Figure 1 is well-positioned, as each edge affinely spans a line of slope m with $|m| \neq 1$, but is not a realisation for a 2-tree decomposition of W_5. The rightmost placement is not well-positioned as the edges incident to v_0 affinely span lines of slope ± 1.

2.3 Graph rigidity

Motivation for considering realisation problems of this type comes from graph rigidity in normed spaces. Consider again a simple graph G and a norm on $\mathbb{R}^d$ of the form (1). A framework (G,p) in $\mathbb{R}^d$ is *(locally) rigid* with respect to this norm if every edge-length preserving continuous motion of the vertices is obtained from an isometric motion of the space. The notion of a well-positioned framework (G,p), introduced above, is equivalent to the condition that the *rigidity map*,

$$f_G : (\mathbb{R}^d)^V \to \mathbb{R}^E, \quad (x(v))_{v \in V} \mapsto (\|x(v) - x(w)\|)_{vw \in E},$$

is differentiable at $p = (p(v))_{v \in V}$. Note that this condition is redundant if the norm under consideration is a smooth norm, since the rigidity map for such norms will always be differentiable. For norms of type (1), the rigidity map f_G fails to be differentiable at $p = (p(v))_{v \in V}$ whenever there is an edge $uv \in E$ and distinct $k, l \in \{1, \ldots, d\}$ such that,

$$\|p(u) - p(v)\| = |(p(u) - p(v)) \cdot \tilde{e}_k| = |(p(u) - p(v)) \cdot \tilde{e}_l|.$$

If (G, p) is well-positioned, then vectors which lie in the kernel of the differential $df_G(p)$ are referred to as *infinitesimal flexes* of (G, p). This kernel will always contain as a subspace the so-called *trivial infinitesimal flexes* of (G, p) which are derived from isometries of the normed space. For norms on $\mathbb{R}^d$ of the form (1), the trivial infinitesimal flexes are precisely the translational vectors of the form $(a, \ldots, a) \in (\mathbb{R}^d)^V$, where $a \in \mathbb{R}^d$. Moreover, it can be shown (see [6]) that local rigidity is equivalent to the condition that (G, p) has no non-trivial infinitesimal flexes. This latter property is known as *infinitesimal rigidity*. For this reason, we will simply refer to a well-positioned framework as being *rigid* whenever it is locally (or equivalently, infinitesimally) rigid. A framework (G, p) is *minimally rigid* (or *isostatic*) if it is rigid and removing any edge from G results in a framework which is not rigid.

Consider an ℓ^q norm on $\mathbb{R}^d$, where $q \in [1, \infty]$ and $q \neq 2$. It is shown in [8] that a necessary condition for a (well-positioned) framework (G, p) on a simple graph G to be isostatic with respect to ℓ^q is that G is (d, d)-tight. It is also shown that in the case $d = 2$, a converse statement holds: every $(2, 2)$-tight simple graph G admits a (well-positioned) framework (G, p) in the plane which is isostatic for the ℓ^q norm. It is conjectured that this equivalence extends to $d \geqslant 3$. In the case of ℓ^∞, the spanning tree characterisation of (d, d)-tight graphs obtained by Nash-Williams [12] and Tutte [16], together with the following result, support this conjecture and provide the link to the realisation problems considered in this article.

Theorem 2.1. [8, Propositions 4.3 & 4.4] Let G be a simple graph and let (G, p) be a well-positioned framework in $(\mathbb{R}^d, \|\cdot\|_\infty)$. The following statements are equivalent:

(a) (G, p) is minimally rigid;

(b) the monochrome subgraphs induced by the framework colouring κ_p are spanning trees in G.

To summarise, if G is a simple graph which is (d, d)-tight then it admits a d-tree decomposition and so, by the above theorem, to show that G admits a well-positioned isostatic framework in $(\mathbb{R}^d, \|\cdot\|_\infty)$ it is sufficient to prove that some d-tree decomposition of G can be realised in $\mathbb{R}^d$.

Example 2. With reference to Figure 1, note that we may regard the first three of these figures as well-positioned bar-joint frameworks in the ℓ^∞-plane. By Theorem 2.1, the first two frameworks are minimally rigid while the third is flexible. A flex of the third framework is obtained, for example, by fixing the vertices v_1, v_2 and applying a horizontal translation to the vertices v_0, v_3, v_4.

2.4 Inductive constructions

Nash-Williams [12] and Tutte [16] independently characterised the multi-graphs which admit a d-tree decomposition as those which are (d, d)-tight:

Theorem 2.2. A multi-graph $G = (V, E)$ is expressible as an edge-disjoint union of d spanning trees if and only if

(a) $|E| = d|V| - d$, and

(b) $i(X) \leqslant d|X| - d$ for all $\emptyset \neq X \subseteq V$.

Let $d \geqslant 2$ and $0 \leqslant j \leqslant d - 1$. A *$d$-dimensional j-extension* of a graph $G = (V, E)$ forms a new graph G' by first deleting some set of edges F from G, with $|F| = j$, and then appending a new vertex v to G, incident to $d + j$ new edges, such that $\mathrm{mult}(N_{G'}(v)) \supseteq \mathrm{mult}(V_G(F))$. Here $\mathrm{mult}(N_{G'}(v))$ denotes the multiset of vertices in $N_{G'}(v)$ in which each vertex w is repeated according to the number of parallel edges vw in G'. Similarly, $\mathrm{mult}(V_G(F))$ denotes the multiset of vertices in $V_G(F)$ in which each vertex w is repeated according to the number of edges in F which are incident to w.

The inverse of a d-dimensional j-extension is a *d-dimensional j-reduction* which, given a graph G', forms G by deleting some vertex v of degree $d+j$ and then adding a set F of j edges between the vertices in $N_{G'}(v)$ such that $\mathrm{mult}(N_{G'}(v)) \supseteq \mathrm{mult}(V_G(F))$. Extensions and reductions of this type were first introduced by Henneberg [3], and so are also known as Henneberg moves and inverse Henneberg moves respectively.

During his work on the rigidity of body-bar frameworks, Tay [14] found the following inductive construction of (d, d)-tight graphs:

Theorem 2.3. A multi-graph $G = (V, E)$ is (d, d)-tight if and only if there exists a sequence of graphs

$$K_1 = G^{(1)} \to G^{(2)} \to \cdots \to G^{(n)} = G$$

such that for all $2 \leqslant i \leqslant n$, $G^{(i)}$ is obtained from $G^{(i-1)}$ by a d-dimensional j-extension, for some $0 \leqslant j \leqslant d - 1$.

Tay later used the inductive construction in Theorem 2.3 to obtain a new proof of Theorem 2.2, see [15]. Combining Theorems 2.2 and 2.3 gives the following result, which is the starting point of this paper:

Theorem 2.4. A multi-graph $G = (V, E)$ is the edge-disjoint union of d spanning trees if and only if there exists a sequence of graphs

$$K_1 = G^{(1)} \to G^{(2)} \to \cdots \to G^{(n)} = G$$

such that for all $2 \leqslant i \leqslant n$, $G^{(i)}$ is obtained from $G^{(i-1)}$ by a d-dimensional j-extension, for some $0 \leqslant j \leqslant d - 1$.

3 Inductive construction for d-tree decompositions

Given a d-tree decomposition $\mathcal{G} = (G; T_1, \ldots, T_d)$, we wish to adapt the construction in Theorem 2.4 so that we simultaneously construct each of the monochrome spanning trees $T_1, T_2, \ldots, T_d$. To do this, we consider an alternative move to the above d-dimensional j-extension, which we instead call a *d-tree j-extension*. This modification is intuitive, however to the best of our knowledge it has not appeared in the literature. For completeness, we prove it here. We first consider a single tree.

Let $j \geqslant 0$, G be a graph and $T \subseteq G$ be a tree with $|E(T)| = j$. A *tree j-extension* of G on T forms a new graph G' by first deleting $E(T)$, and then appending a new vertex v to G, incident to $j+1$ new edges, such that $N_{G'}(v) = V_G(T)$. The inverse of a tree j-extension is a *tree j-reduction.*

Lemma 3.1. Let $T = (V, E)$ be a simple graph, and let $S = (U, F)$ be a subgraph of T which is a tree. Suppose $T' = (V', E')$ is formed from T by a tree $|F|$-extension on F. Then T is a tree if and only if T' is a tree.

Proof. By the definition of a tree $|F|$-extension, $|E| = |E'| - 1$ and $|V| = |V'| - 1$. It follows that $|E'| = |V'| - 1$ if and only if $|E| = |V| - 1$. So to prove that T is a tree if and only if T' is a tree, it remains to note that T is connected if and only if T' is connected. $\square$

It is easy to see that d-tree decompositions satisfy the following fact.

Lemma 3.2. Let G be a multi-graph, with $|V(G)| \geqslant 2$, which admits a d-tree decomposition for some $d \geqslant 1$. Then $d \leqslant \delta(G) \leqslant 2d - 1$.

This lemma implies that the following definition of a d-tree j-extension, need only consider $0 \leqslant j \leqslant d - 1$.

Definition 3.3. A d-tree decomposition $\mathcal{G}' = (G'; T'_1, \ldots, T'_d)$ is said to be obtained from a d-tree decomposition $\mathcal{G} = (G; T_1, \ldots, T_d)$ by a *d-tree j-extension* if

(a) $0 \leqslant j \leqslant d - 1$,

(b) G' is obtained from G by deleting j edges and appending a vertex incident to $d + j$ edges, and,

(c) for each $1 \leqslant i \leqslant d$, T'_i is obtained from T_i by a tree k_i-extension, for some $0 \leqslant k_i \leqslant d - 1$.

Note that $\sum_{i=1}^{d} k_i = j$, and that Lemma 3.1 ensures that $\mathcal{G}'$ is indeed a d-tree decomposition whenever $\mathcal{G}$ is.

Proposition 3.4. Let $\mathcal{G}' \in \mathcal{G}_d$ be a d-tree decomposition with $\mathcal{G}' \neq \mathcal{K}_1$. Then there exists a d-tree decomposition $\mathcal{G} \in \mathcal{G}_d$ such that $\mathcal{G}'$ is a d-tree j-extension of $\mathcal{G}$.

Proof. Let $\mathcal{G}' = (G'; T_1', \ldots, T_d')$ where $G' = (V', E')$ and $T_i' = (V', E_i')$ for each $i = 1, \ldots, d$. By Lemma 3.2, there exists $v \in V'$ such that $d \leqslant d_{G'}(v) \leqslant 2d - 1$. For each tree T_i', perform a tree $|F_i|$-reduction at v which forms the graph $T_i = (V' - v, E_i)$ by deleting v and adding a set of edges F_i between the vertices in $N_{T_i'}(v)$ such that $(N_{T_i'}(v), F_i)$ is a tree.

Let $G = (V' - v, E_1 \cup \cdots \cup E_d)$. By Lemma 3.1, $T_1, \ldots, T_d$ are trees and so $\mathcal{G} = (G; T_1, \ldots, T_d)$ is a d-tree decomposition. In order for the move which formed $\mathcal{G}$ from $\mathcal{G}'$ to be a d-tree j-reduction, it only remains to show that $0 \leqslant |F_i| \leqslant d - 1$ for all $1 \leqslant i \leqslant d$.

Since $T_1', T_2', \ldots, T_d'$ are spanning trees of G', we know $d_{T_i'}(v) \geqslant 1$ for all $1 \leqslant i \leqslant d$. Further, by our choice of v,

$$\sum_{i=1}^{d} d_{T_i'}(v) = d_{G'}(v) \leqslant 2d - 1.$$

Hence $1 \leqslant d_{T_i'}(v) \leqslant d$ for all $1 \leqslant i \leqslant d$. By the definition of a tree $|F_i|$-reduction, $|F_i| = d_{T_i'}(v) - 1$. And so $0 \leqslant |F_i| \leqslant d - 1$, as required. □

This gives the sought alternative to Tay's construction.

Corollary 3.5. Let $\mathcal{G} = (G; T_1, \ldots, T_d)$ be a d-tree decomposition. Then, there exists a sequence of d-tree decompositions

$$\mathcal{K}_1 = \mathcal{G}^{(1)} \to \mathcal{G}^{(2)} \to \cdots \to \mathcal{G}^{(n)} = \mathcal{G}$$

such that for all $2 \leqslant i \leqslant n$, $\mathcal{G}^{(i)}$ is obtained from $\mathcal{G}^{(i-1)}$ by a d-tree j-extension, for some $0 \leqslant j \leqslant d - 1$.

Note that both d-dimensional j-extensions and d-tree j-extensions are moves which delete j edges and append a vertex of degree $d + j$. In general, these two graph moves are distinct. However, when $j \in \{0, 1\}$, they coincide.

4 Realisations for 2-tree decompositions

In this section, we apply our inductive construction from Corollary 3.5 to show that every 2-tree decomposition has a plane realisation. To do this, we first extend the definitions of a well-positioned framework (G, p) in $\mathbb{R}^d$ and an induced framework colouring κ_p, which were given in Section 2.2, to accommodate multi-graphs.

Let $G = (V, E)$ be a multi-graph with no loops and let $p : V \to \mathbb{R}^d$ be an injective map. As before we refer to the pair (G, p) as a *framework* in $\mathbb{R}^d$. Suppose a pair of vertices $u, v \in V$ are joined by exactly t edges, where $1 \leqslant t \leqslant d$. The pair $\{p(u), p(v)\}$ is said to be *well-positioned* if there exist exactly t distinct elements $j_1, \ldots, j_t \in \{1, \ldots, d\}$ such that

$$\|p(u) - p(v)\|_\infty = |(p(u) - p(v)) \cdot e_{j_k}|$$

for $k = 1, 2, \ldots, t$. We refer to $j_1, \ldots, j_t$ as the *framework colours* for the pair $\{p(u), p(v)\}$. The framework (G, p), on the multi-graph G, is said to be *well-positioned* if, for every edge $uv \in E(G)$, the pair $\{p(u), p(v)\}$ is well-positioned.

When a framework (G,p) is well-positioned, the multi-graph G inherits an edge-labelling $\kappa_p : E(G) \to \{1,2,\dots,d\}$ whereby each of the t edges connecting a pair of vertices u and v is assigned one of the distinct framework colours $j_1,\dots,j_t$ for the pair $\{p(u),p(v)\}$. The edge-labelling κ_p is referred to as a *framework colouring* for (G,p). Note that this framework colouring is unique up to permutation of colours between parallel edges. Such permutations will not create any problems in what follows. The subgraph $G[\kappa_p^{-1}(k)]$ spanned by edges with framework colour k is again referred to as a *monochrome subgraph* of G.

Given a d-tree decomposition $\mathcal{G} = (G; T_1,\dots,T_d)$ of a multi-graph G, a *realisation* of $\mathcal{G}$ is a framework (G,p) in $\mathbb{R}^d$ with the property that $T_1,\dots,T_d$ are the induced monochrome subgraphs of G, for some choice of framework colouring κ_p. Note that by relabelling $T_1,\dots,T_d$ we may assume, without loss of generality, that for $i = 1,\dots,d$, edges uv in T_i satisfy,

$$\|p(u)-p(v)\|_\infty = |(p(u)-p(v))\cdot e_i|.$$

Proposition 4.1. Let $\mathcal{G} = (G;T_1,T_2)$ be a 2-tree decomposition and suppose that $\mathcal{G}' = (G';T_1',T_2')$ is a 2-tree decomposition which is obtained by applying a 2-tree 0-extension to $\mathcal{G}$.

If $\mathcal{G}$ has a realisation p in the plane then $\mathcal{G}'$ has a realisation p' in the plane with the property that $p'(w) = p(w)$ for all $w \in V(G)$.

Proof. Suppose the 2-dimensional 0-reduction which forms G from G' deletes the vertex v. By the definition of a 2-dimensional 0-reduction, $d_{G'}(v) = 2$, and G is formed from G' by deleting v and the two edges incident to v. Since T_1' and T_2' are both spanning trees of G', this implies that v is a leaf node of both T_1' and T_2'. Hence $T_i = T_i' - v$ for $i \in \{1,2\}$. Let $N_{G'}(v) = \{x,y\}$ where x and y may or may not be distinct. Without loss of generality, suppose $vx \in E(T_1')$ and $vy \in E(T_2')$. Since $G = G' - v$, we know that $(G'-v,p)$ is well-positioned. So let $p'(w) = p(w)$ for all $w \in V(G)$. For (G',p') to be well-positioned, it only remains to find a position for $p'(v)$ such that the pairs $\{p'(v),p'(x)\}$ and $\{p'(v),p'(y)\}$ are well-positioned.

If $x \neq y$ then $p'(x) \neq p'(y)$. If we place v within a sufficiently small distance of the intersection of the lines $p'(x)+\lambda(1,0)$ and $p'(y)+\mu(0,1)$, then $\{p'(v),p'(x)\}$ and $\{p'(v),p'(y)\}$ will have framework colours 1 and 2 respectively.

Now suppose $x = y$. Then v is met by a double-edge; one edge in T_1' and the other in T_2'. To satisfy the constraints induced by this colouring, we must place $p'(v)$ such that

$$\|p'(v)-p'(x)\|_\infty = |p'(v)_1 - p'(x)_1| = |p'(v)_2 - p'(x)_2|.$$

If we place $p'(v)$ at any point of the lines $p'(x)+\lambda(1,1)$ or $p'(x)+\lambda(1,-1)$, then we will satisfy these constraints, and, since vx is a double-edge, the pair $\{p'(v),p'(x)\}$ will be well-positioned.

In both cases we can choose $p'(v)$ so that it is not coincident with another vertex of G'. Moreover, (G',p') is well-positioned, and the induced monochrome subgraphs are T_1' and T_2'. $\square$

To complete the inductive construction we require a geometric method of realising 2-tree 1-extensions which accommodates parallel edges.

Proposition 4.2. Let $\mathcal{G} = (G; T_1, T_2)$ be a 2-tree decomposition and suppose that $\mathcal{G}' = (G'; T_1', T_2')$ is a 2-tree decomposition which is obtained by applying a 2-tree 1-extension to $\mathcal{G}$.

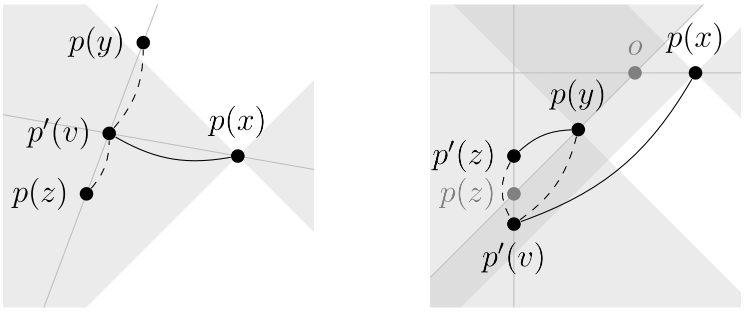
If $\mathcal{G}$, and every 2-tree decomposition with fewer vertices than $\mathcal{G}$, has a realisation p in the plane then $\mathcal{G}'$ has a realisation p' in the plane.

Proof. Suppose the 2-dimensional 1-reduction which forms G from G' deletes the vertex v. By the definition of a 2-dimensional 1-reduction, $d_{G'}(v) = 3$, and G is formed from G' by deleting v and all three edges incident to v, before adding a single edge between the vertices in $N_{G'}(v)$. Since T_1' and T_2' are both spanning trees of G', this implies that v is a leaf node of one of these trees, and is incident to two edges of the other tree. Without loss of generality assume $d_{T_1'}(v) = 1$ and $d_{T_2'}(v) = 2$. Let $N_{T_1'}(v) = \{x\}$ and $N_{T_2'}(v) = \{y, z\}$, where $y \notin \{x, z\}$, but potentially $x = z$. Then, $T_1 = T_1' - v$. In order for T_2 to be a tree, it must be formed by deleting v from T_2', and then adding an edge e between y and z. In other words, $T_2 = T_2' - v + e$. Let (G, p) be a realisation for $\mathcal{G}$ in the plane. We shall use p to construct a realisation p' for $\mathcal{G}'$.

For all pairs of points $(a, b), (c, d) \in \mathbb{R}^2$, write $(a, b) \leqslant (c, d)$ if $a \leqslant c$ and $b \leqslant d$ and write $(a, b) < (c, d)$ if, in addition, $(a, b) \neq (c, d)$. Note that reflecting the set of points $\{p(w) : w \in V(G)\}$ through either of the coordinate axes in $\mathbb{R}^2$ generates another realisation of $\mathcal{G}$ in which the framework colours for all pairs $\{p(u), p(w)\}$ are preserved. For a pair of vertices $u, w \in V(G)$, write $u \sim w$ if either $u = w$, or, w is joined to u by a sequence of parallel edges in $G - e$.

Case 1. Suppose $x \neq z$ and $yz \notin E(T_1)$.
By reflecting in a coordinate axis we may assume, without loss of generality, that $p(z) < p(y)$. Let $p'(w) = p(w)$ for all $w \in V(G)$. Since $yz \in E(T_2)$, any placement of $p'(v)$ on the line $p'(y) + \mu(p'(z) - p'(y))$, $\mu \in \mathbb{R}$, which is distinct from $p'(y)$ and $p'(z)$, will ensure the pairs $\{p'(v), p'(y)\}$ and $\{p'(v), p'(z)\}$ are well-positioned with framework colour 2. Similarly, for all $-1 < a < 1$, any placement of $p'(v)$ on the line $p'(x) + \lambda(1, a)$, $\lambda \in \mathbb{R}$, which is distinct from $p'(x)$, will ensure $\{p'(v), p'(x)\}$ is well-positioned with framework colour 1. Hence, if we place $p'(v)$ in a small neighbourhood of the intersection of the two lines $p'(x) + \lambda(1, a)$ and $p'(y) + \mu(p'(z) - p'(y))$, and $p'(v)$ is chosen so that it is not coincident with any other vertex of G', then (G', p') is a realisation of $\mathcal{G}'$. See Figure 2(a).

Case 2. Suppose $x \neq z$ and $yz \in E(T_1)$.
Contract the parallel edges between y and z in G to form a new 2-tree decomposition $\mathcal{G}^* = (G^*; T_1^*, T_2^*)$. Write $V(G^*) = V(G) - \{y, z\} + \{w_0\}$ where w_0 is the vertex obtained by identifying y and z. Since $\mathcal{G}^*$ has fewer vertices than $\mathcal{G}$, there exists a realisation p^* for $\mathcal{G}^*$ in the plane. By reflecting in a coordinate axis we may assume, without loss of generality, that $p^*(w_0) < p^*(x)$. From this realisation, we can construct a new realisation p for $\mathcal{G}$ with the property that $p(z) < p(y) < p(x)$. Formally, set $p(z) = p^*(w_0)$, $p(y) = p^*(w_0) + \epsilon'(1, 1)$, $p(w) = p^*(w) + \epsilon'(1, 1)$ for all $w \sim y$ with $w \neq y$, for some sufficiently small $\epsilon' > 0$, and $p(w) = p^*(w)$ for all remaining $w \in V(G)$. Let o denote the intersection of the lines



(a) When $x \neq z$ and $yz \notin E(T_1)$.

(b) When $x \neq z$, $yz \in E(T_1)$ and $o < p(x)$.

Figure 2: Some choices for $p'(v)$ in (G', p') when x, y and z are distinct. Features of (G', p') are shown in black, with solid and dashed black lines denoting edges in T_1' and T_2' respectively. For each vertex $w \in \{x, y, z\}$ we highlight in grey either a shaded region or a line, such that any placement $p'(v)$ within this region ensures the pair $\{p'(v), p'(w)\}$ satisfies the geometric constraints implied by the 2-tree decomposition of $\mathcal{G}'$. Note that in example (b), both T_1 and T_2 contain a yz-edge, so to ensure the pair $\{p'(y), p'(z)\}$ is well-positioned after $yz \in E(T_2)$ is deleted, we shift $p(z)$ upwards.

$p(x) + \lambda(1, 0)$ and $p(y) + \mu(1, 1)$. Note $p(z)$, $p(y)$ and o are collinear and by choosing ϵ' sufficiently small we may assume, without loss of generality, that $p(y) \neq o$.

Let $\epsilon > 0$ and $\delta > 0$. If $o < p(x)$ then, informally, place $p'(v)$ below $p(z)$ and then translate $p(z)$ upwards. See Figure 2(b). Any vertex in G which is joined to z by a sequence of parallel edges in $G - e$ must also be translated upwards. Formally, define p' by setting $p'(v) = p(z) + \delta(0, -1)$ and, for each $w \in V(G)$,

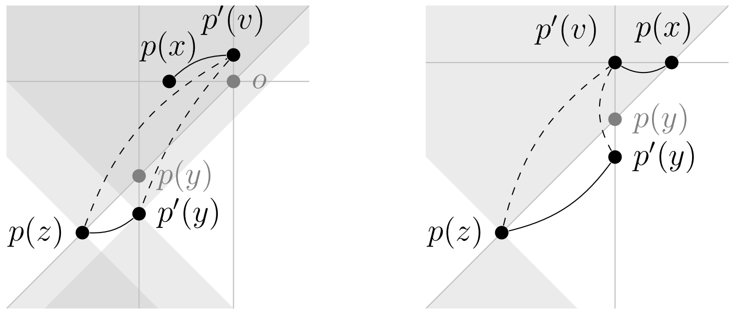
$$p'(w) = \begin{cases} p(w) + \epsilon(0, 1) & \text{if } w \sim z, \text{ and,} \\ p(w) & \text{otherwise.} \end{cases}$$

If $p(x) < o$ then, informally, place $p'(v)$ above o and then translate $p(y)$ downwards. See Figure 3(a). Any vertex in G which is joined to y by a sequence of parallel edges in $G - e$ must also be translated downwards. Formally, define p' by setting $p'(v) = o + \delta(0, 1)$ and, for each $w \in V(G)$,

$$p'(w) = \begin{cases} p(w) + \epsilon(0, -1) & \text{if } w \sim y, \text{ and,} \\ p(w) & \text{otherwise.} \end{cases}$$

If $p(x) = o$ then $p(x)$, $p(y)$ and $p(z)$ are collinear. Choose $p'(v)$ in a small neighbourhood of the intersection of the horizontal line through $p(x)$ and the vertical line through $p(y)$. Then translate $p(y)$ downwards. See Figure 3(b). Any vertex in G which is joined to y by a sequence of parallel edges in $G - e$ must also be translated downwards. Thus, for each $w \in V(G)$ we define,

$$p'(w) = \begin{cases} p(w) + \epsilon(0, -1) & \text{if } w \sim y, \text{ and,} \\ p(w) & \text{otherwise.} \end{cases}$$



(a) When $x \neq z$, $yz \in E(T_1)$ and $p(x) < o$.

(b) When $x \neq z$, $yz \in E(T_1)$ and $p(x) = o$.

Figure 3: Further choices for $p'(v)$ in (G', p') when x, y and z are distinct and $yz \in E(T_1)$. In each case, we place $p'(y)$ below $p(y)$ to ensure the pair $\{p'(y), p'(z)\}$ is well-positioned after $yz \in E(T_2)$ is deleted.

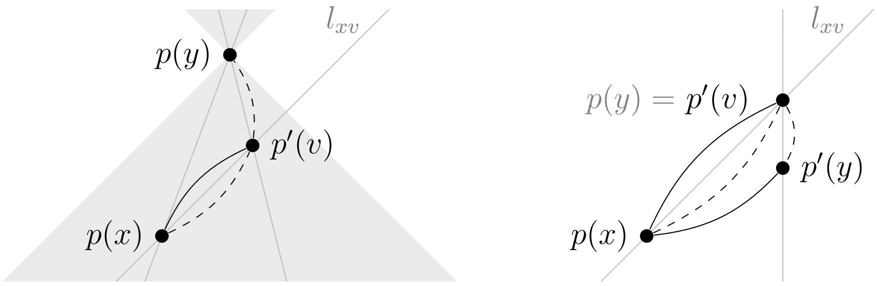
Case 3. Suppose $x = z$.

In this case v sends a double edge to x: one edge in T_1' and the other in T_2'. If $xy \notin E(T_1)$, then let $p'(w) = p(w)$ for all $w \in V(G)$. Place $p'(v)$ at the intersection of the lines $p'(x) + \lambda(1, 1)$ and $p'(y) + \mu(a, 1)$ where $-1 < a < 1$. Choose a such that $p'(v)$ is not coincident with any other vertex of (G', p'). See Figure 4(a).

If $xy \in E(T_1)$ then by reflecting in a coordinate axis we may assume, without loss of generality, that $p(x) < p(y)$. Informally, place $p'(v)$ at $p(y)$ and then translate $p(y)$ downwards. Any vertex in G which is joined to y by a sequence of parallel edges in $G - e$ must also be translated downwards. Formally, define p' by setting $p'(v) = p(y)$ and, for each $w \in V(G)$,

$$p'(w) = \begin{cases} p(w) + \epsilon(0, -1) & \text{if } w \sim y, \text{ and,} \\ p(w) & \text{otherwise,} \end{cases}$$

where $\epsilon > 0$. See Figure 4(b)



(a) When $x = z$ and $xy \notin E(T_1)$.

(b) When $x = z$ and $xy \in E(T_1)$.

Figure 4: Choices for $p'(v)$ in (G', p') when $x = z$. Since both T_1' and T_2' contain an xv-edge, the line l_{xv} through $p'(x)$ and $p'(v)$ has slope 1.

In each case, ϵ and δ can be chosen sufficiently small so that $p'(v)$ satisfies the required constraints and so that the points $\{p'(w) : w \in V(G')\}$ are distinct. Thus (G', p') is a realisation for $\mathcal{G}'$. $\square$

We can now prove that every 2-tree decomposition has a realisation in the plane.

Theorem 4.3. Let $\mathcal{G} = (G; T_1, T_2)$ be a 2-tree decomposition. Then there exists a realisation for $\mathcal{G}$ in the plane.

Proof. By Corollary 3.5, there exists a sequence of 2-tree decompositions and 2-tree 0 and 1-extensions which construct $\mathcal{G}$ from the base element $\mathcal{K}_1$. Note that a realisation of $\mathcal{K}_1$ is obtained by placing the vertex of $\mathcal{K}_1$ anywhere in the plane. By Propositions 4.1 and 4.2, there exists a realisation for every 2-tree decomposition in this sequence, in particular, such a realisation exists for $\mathcal{G}$. $\square$

As a corollary we obtain an alternative proof of the following result from [8].

Corollary 4.4. Let G be a $(2, 2)$-tight simple graph. Then there exists a placement p such that (G, p) is well-positioned and minimally rigid in $(\mathbb{R}^2, \|\cdot\|_\infty)$.

Proof. By Theorem 2.2, G admits a 2-tree decomposition $\mathcal{G} = (G; T_1, T_2)$. By Theorem 4.3, this 2-tree decomposition has a realisation (G, p) in the plane. By Theorem 2.1, this realisation is minimally rigid in $(\mathbb{R}^2, \|\cdot\|_\infty)$. $\square$

5 Symmetric 2-tree decompositions

In this section we adapt the methods of the previous pages to show that every *symmetric* 2-tree decomposition, with no fixed edges, can be realised as a symmetric framework in the plane which is minimally rigid with respect to ℓ^1 or ℓ^∞ distance constraints. We focus on frameworks with reflectional symmetry through a coordinate axis. Our motivation comes from recent work ([9]) which characterises the class of symmetric graphs which admit a symmetric and minimally rigid realisation in the plane, in terms of certain symmetric 2-tree decompositions. The main results of this section make the task of constructing examples of symmetric minimally rigid frameworks significantly easier. These results also suggest that further adaptations may be possible in other contexts, such as gain graph constructions for symmetric frameworks.

By a $\mathbb{Z}_2$*-symmetric multi-graph* we will mean a pair (G, θ) consisting of a multi-graph G, with no loops, and a non-trivial group homomorphism $\theta : \mathbb{Z}_2 \to \mathrm{Aut}(G)$. Let $\mathbb{Z}_2 = \langle s \rangle$. To simplify notation, we denote $\theta(s)$ by s_θ and for each edge $e = v_1v_2$ we write $s_\theta(e) = s_\theta(v_1)s_\theta(v_2)$. The *vertex orbit* of $v \in V(G)$ is the set $\{v, s_\theta(v)\}$ and the *edge orbit* of $e \in E(G)$ is $\{e, s_\theta(e)\}$. We say that a vertex v (respectively an edge e) is *fixed* if $v = s_\theta(v)$ (respectively $e = s_\theta(e)$).

Definition 5.1. A *symmetric* 2*-tree decomposition* is a tuple $\mathcal{G} = (G; T_1, T_2; \theta)$ such that,

(a) $(G; T_1, T_2)$ is a 2-tree decomposition,

(b) (G, θ) is a $\mathbb{Z}_2$-symmetric multi-graph, and

(c) $s_\theta(T_1) = T_1$ and $s_\theta(T_2) = T_2$.

Denote by $\mathcal{G}_2^{sym}$ the set of all symmetric 2-tree decompositions with no fixed edges. We formally include in $\mathcal{G}_2^{sym}$ the tuple $\mathcal{K}_1 = (K_1; T_1, T_2; \theta)$ where K_1 is the graph with a single vertex v_0 and no edges, T_1 and T_2 have empty edge set, and $\theta : \mathbb{Z}_2 \to \mathrm{Aut}(K_1)$ is the trivial group homomorphism with $s_\theta(v_0) = v_0$. This tuple will form the base element of a construction scheme for $\mathcal{G}_2^{sym}$.

Lemma 5.2. Let $\mathcal{G} = (G; T_1, T_2; \theta) \in \mathcal{G}_2^{sym}$ with $\mathcal{G} \neq \mathcal{K}_1$. Then s_θ fixes exactly one vertex which has even degree at least 4.

Proof. The case where G is a simple graph is proved in [9, Lemma 3] and this proof extends to the multi-graph case. □

5.1 Multi-graph construction scheme for $\mathcal{G}_2^{sym}$

The following two graph moves were applied in the context of simple graphs, and with $d = 2$, in [9] (where they were referred to as $\mathbb{Z}_2$-symmetric 1-extensions and $\mathbb{Z}_2$-symmetric 2-extensions). Here we will first extend the moves to multi-graphs, allowing $d \in \{1, 2\}$, and then introduce corresponding moves for symmetric 2-tree decompositions.

Definition 5.3. A $\mathbb{Z}_2$-symmetric multi-graph (G', θ') is said to be obtained from a $\mathbb{Z}_2$-symmetric multi-graph (G, θ) by a *symmetric d-dimensional 0-extension* if,

(a) $V(G') = V(G) \cup \{v, s_{\theta'}(v)\}$ where $v, s_{\theta'}(v) \notin V(G)$ and $v \neq s_{\theta'}(v)$,

(b) $s_{\theta'}|_{V(G)} = s_\theta$,

(c) $E(G') = E(G) + \{vv_i, s_{\theta'}(vv_i) : i = 1, \ldots, d\}$ for some $v_1, \ldots, v_d \in V(G)$ not necessarily distinct.

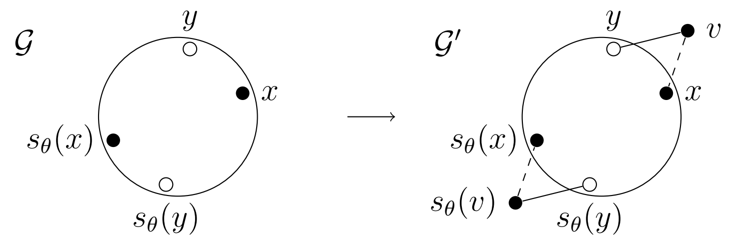
See Figure 5 for examples of such a move when $d = 2$.

Definition 5.4. A $\mathbb{Z}_2$-symmetric multi-graph (G', θ') is said to be obtained from a $\mathbb{Z}_2$-symmetric multi-graph (G, θ) by a *symmetric d-dimensional 1-extension* if,

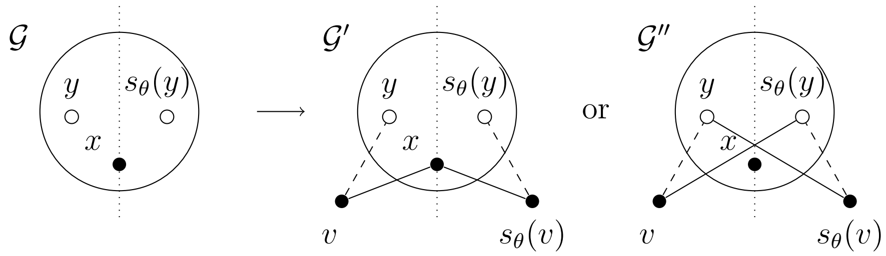
(a) $V(G') = V(G) \cup \{v, s_{\theta'}(v)\}$ where $v, s_{\theta'}(v) \notin V(G)$ and $v \neq s_{\theta'}(v)$,

(b) $s_{\theta'}|_{V(G)} = s_\theta$,

(c) there exist $d + 1$ vertices $v_1, \ldots, v_{d+1} \in V(G)$, with $e = v_1v_2 \in E(G)$ but which are otherwise not necessarily distinct, such that $E(G') = E(G) - \{e, s_\theta(e)\} + \{vv_i, s_{\theta'}(vv_i) : i = 1, \ldots, d + 1\}$.

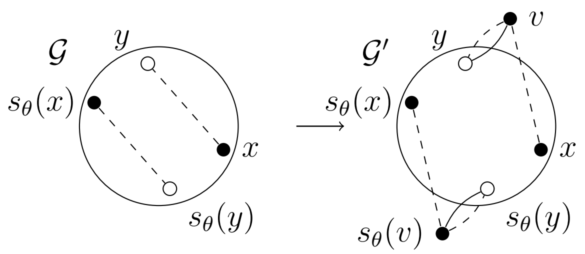


(a) A symmetric 2-tree decomposition $\mathcal{G}'$ formed from a symmetric 2-tree decomposition $\mathcal{G}$ by a symmetric 2-tree 0-extension.

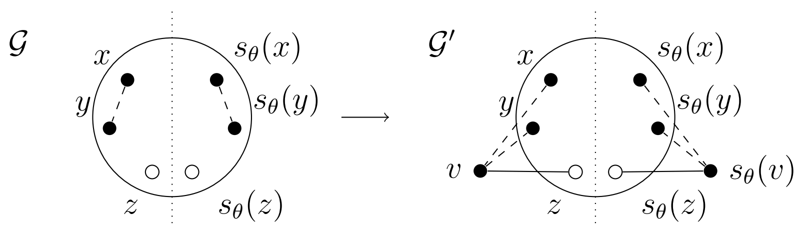


(b) Two symmetric 2-tree decompositions, $\mathcal{G}'$ and $\mathcal{G}''$, each formed by a symmetric 2-tree 0-extension on the symmetric 2-tree decomposition $\mathcal{G}$.

Figure 5: Symmetric 2-tree 0-extensions under half-turn (a) and single mirror (b) symmetry.



(a) Half-turn symmetry



(b) Single mirror symmetry

Figure 6: Symmetric 2-tree 1-extensions. In each case, the symmetric 2-tree decomposition $\mathcal{G}'$ is obtained from the symmetric 2-tree decomposition $\mathcal{G}$ by a symmetric 2-tree 1-extension.

See Figure 6 for examples when $d = 2$.

We now adapt the above symmetric graph moves to incorporate symmetric 2-tree decompositions. Again see Figures 5 and 6 for illustrations of these moves.

Definition 5.5. A symmetric 2-tree decomposition $\mathcal{G}' = (G'; T_1', T_2'; \theta')$ is said to be obtained from a symmetric 2-tree decomposition $\mathcal{G} = (G; T_1, T_2; \theta)$ by a *symmetric 2-tree j-extension*, where $j \in \{0, 1\}$, if

(a) (G', θ') is obtained from (G, θ) by a symmetric 2-dimensional j-extension, and,

(b) for $i \in \{1, 2\}$, (T_i', θ') is obtained from (T_i, θ) by a symmetric k_i-extension, for some $k_i \in \{0, 1\}$ where $k_1 + k_2 = j$.

We now prove the existence of a construction scheme for symmetric 2-tree decompositions in $\mathcal{G}_2^{sym}$ which uses symmetric 2-tree 0 and 1-extensions.

Theorem 5.6. Let $\mathcal{G}' = (G'; T_1', T_2'; \theta') \in \mathcal{G}_2^{sym}$ with $\mathcal{G}' \neq \mathcal{K}_1$. Then there exists $\mathcal{G} = (G; T_1, T_2; \theta) \in \mathcal{G}_2^{sym}$ such that $\mathcal{G}'$ is obtained from $\mathcal{G}$ by either a symmetric 2-tree 0-extension or a symmetric 2-tree 1-extension.

Proof. By Lemma 3.2, G' has a vertex v of degree 2 or 3, and by Lemma 5.2, this vertex is not fixed. Hence the vertex orbit $\{v, s_{\theta'}(v)\}$ contains two distinct vertices with $d_{G'}(v) = d_{G'}(s_{\theta'}(v))$. Since G' is the edge-disjoint union of spanning trees T_1' and T_2', v is incident to at least one edge $e_i \in E(T_i')$ from each of these trees.

Claim 3. *If $d_{G'}(v) = 2$, then $\mathcal{G}'$ is formed from a symmetric 2-tree decomposition $\mathcal{G}$ by a symmetric 2-tree 0-extension.*

Proof. In this case $N_{G'}(v) = \{u_1, u_2\}$, where u_1 and u_2 need not be distinct. Since $d_{G'}(v) = 2$, and $s_{\theta'}(v) \notin N_{G'}(v)$, we can remove both v and $s_{\theta'}(v)$ from G' by separate 2-dimensional 0-reductions to obtain a $\mathbb{Z}_2$-symmetric multi-graph (G, θ) with $s_\theta = s_{\theta'}|_{V(G)}$. Let $T_i = T_i' - \{v, s_{\theta'}(v)\}$ and note that $s_\theta(T_i) = T_i$. Thus $\mathcal{G} = (G; T_1, T_2; \theta) \in \mathcal{G}_2^{sym}$. By reversing this process, $\mathcal{G}'$ may be obtained from $\mathcal{G}$ by a symmetric 2-tree 0-extension. □

Claim 4. *If $d_{G'}(v) = 3$ then $\mathcal{G}'$ is formed from a symmetric 2-tree decomposition $\mathcal{G}$ by a symmetric 2-tree 1-extension.*

Proof. In this case v is incident to a third edge e_3. Without loss of generality, suppose $e_3 \in E(T_1')$. For $i \in \{1, 2, 3\}$, none of the edges e_i are fixed, so they each terminate at some $u_i \in V(G') - \{v, s_{\theta'}(v)\}$. Hence each edge $s_{\theta'}(e_i)$ incident to $s_{\theta'}(v)$ terminates at $s_{\theta'}(u_i) \in V(G') - \{v, s_{\theta'}(v)\}$.

Since $e_1, e_3 \in E(T_1')$ and T_1' is a tree, $u_1u_3 \notin E(T_1')$, and so, since T_1' is symmetric under θ', $s_{\theta'}(u_1)s_{\theta'}(u_3) \notin E(T_1')$ either. If $\{u_1, u_3\} = \{s_{\theta'}(u_1), s_{\theta'}(u_3)\}$ then $v, u_1, s_{\theta'}(v), u_3, v$ is a cycle in T_1', which is a contradiction. Hence $\{u_1, u_3\} \neq \{s_{\theta'}(u_1), s_{\theta'}(u_3)\}$ and so we can perform a 2-dimensional 1-reduction at v which adds the edge u_1u_3, followed by a 2-dimensional 1-reduction at $s_{\theta'}(v)$ which adds the edge $s_{\theta'}(u_1)s_{\theta'}(u_3)$ to form the graph G.

Let $\mathcal{G} = (G; T_1, T_2; \theta)$ be the symmetric 2-tree decomposition with,

- $G = G' - \{v, s_{\theta'}(v)\} + \{u_1u_3, s_{\theta'}(u_1u_3)\}$,
- $T_1 = T_1' - \{v, s_{\theta'}(v)\} + \{u_1u_3, s_{\theta'}(u_1u_3)\}$,
- $T_2 = T_2' - \{v, s_{\theta'}(v)\}$, and,
- $s_\theta = s_{\theta'}|_{V(G)}$.

Then $s_\theta(T_i) = T_i$ and so $\mathcal{G} \in \mathcal{G}_2^{sym}$. Further T_1 was obtained from T_1' by a pair of tree 1-reductions, and T_2 was formed from T_2' by a pair of tree 0-reductions. Thus we can reconstruct $\mathcal{G}'$ from $\mathcal{G}$ by a symmetric 2-tree 1-extension. □

Since $d_{G'}(v) \leqslant 3$, these two claims complete the proof. □

This result implies the inductive constructions sought:

Corollary 5.7. Let $\mathcal{G} = (G; T_1, T_2; \theta) \in \mathcal{G}_2^{sym}$. Then, there exists a sequence of symmetric 2-tree decompositions in $\mathcal{G}_2^{sym}$,

$$\mathcal{K}_1 = \mathcal{G}^{(1)} \to \mathcal{G}^{(2)} \to \cdots \to \mathcal{G}^{(n)} = \mathcal{G}$$

such that for all $2 \leqslant i \leqslant n$, $\mathcal{G}^{(i)}$ is obtained from $\mathcal{G}^{(i-1)}$ by a symmetric 2-tree j-extension, for some $j \in \{0, 1\}$.

5.2 Realisations with $\mathcal{C}_s$-symmetry

We shall now show how to construct a symmetric realisation for any symmetric 2-tree decomposition from the class $\mathcal{G}_2^{sym}$. We prove this explicitly for realisations under reflection symmetry. The argument for half-turn symmetry is similar.

A *symmetric placement* of a $\mathbb{Z}_2$-symmetric multi-graph (G, θ) in the plane is a pair (p, τ) consisting of an injective map $p : V(G) \to \mathbb{R}^2$ and a representation $\tau : \mathbb{Z}_2 \to \mathrm{GL}(\mathbb{R}^2)$ such that $\tau(s)(p(v)) = p(s_\theta(v))$ for all $v \in V(G)$. If $\tau(s)$ is a reflection in a coordinate axis then we refer to the pair (p, τ) as a *$\mathcal{C}_s$-placement* of (G, θ). If $\tau(s)$ is a half-turn rotation about the origin then we refer to (p, τ) as a *$\mathcal{C}_2$-placement* of (G, θ).

A *$\mathcal{C}_s$-realisation* (respectively, *$\mathcal{C}_2$-realisation*) for a symmetric 2-tree decomposition $\mathcal{G} = (G; T_1, T_2; \theta)$ in the plane is a $\mathcal{C}_s$-placement (respectively, *$\mathcal{C}_2$-placement*) (p, τ) of (G, θ) with the property that (G, p) is a realisation for the 2-tree decomposition $(G; T_1, T_2)$.

Proposition 5.8. Let $\mathcal{G} = (G; T_1, T_2; \theta)$ and $\mathcal{G}' = (G'; T_1', T_2'; \theta')$ be a pair of symmetric 2-tree decompositions and suppose $\mathcal{G}'$ is obtained by applying a symmetric 2-tree 0-extension to $\mathcal{G}$.

If $\mathcal{G}$ has a $\mathcal{C}_s$-realisation (p, τ) in the plane then $\mathcal{G}'$ has a $\mathcal{C}_s$-realisation (p', τ) in the plane with the property that $p'(w) = p(w)$ for all $w \in V(G)$.

Proof. Let (p, τ) be a $\mathcal{C}_s$-realisation for $\mathcal{G}$ in the plane. Suppose the symmetric 2-tree 0-extension which forms $\mathcal{G}'$ from $\mathcal{G}$ adjoins the vertices v and $s_{\theta'}(v)$ to G. Let $\mathcal{G}''$ be the intermediate (and non-symmetric) 2-tree decomposition obtained by deleting $s_{\theta'}(v)$ and its incident edges from G'. Note that $\mathcal{G}''$ is obtained from $\mathcal{G}$ by a (non-symmetric) 2-tree

0-extension. By Proposition 4.1, there exists a realisation p'' for $\mathcal{G}''$ with the property that $p''(w) = p(w)$ for all $w \in V(G)$. Define $p'(w) = p(w)$ for all $w \in V(G)$, $p'(v) = p''(v)$ and $p'(s_{\theta'}(v)) = \tau(s)(p'(v))$. The resulting pair (p', τ) is a $\mathcal{C}_s$-realisation for $\mathcal{G}'$. $\square$

To complete the inductive construction we now consider geometric placements for symmetric 2-tree 1-extensions.

Proposition 5.9. Let $\mathcal{G} = (G; T_1, T_2; \theta)$ and $\mathcal{G}' = (G'; T_1', T_2'; \theta')$ be a pair of symmetric 2-tree decompositions and suppose $\mathcal{G}'$ is obtained by applying a symmetric 2-tree 1-extension to $\mathcal{G}$.

If $\mathcal{G}$, and every symmetric 2-tree decomposition with fewer vertices than $\mathcal{G}$, has a $\mathcal{C}_s$-realisation in the plane then $\mathcal{G}'$ has a $\mathcal{C}_s$-realisation in the plane.

Proof. Let (p, τ) be a $\mathcal{C}_s$-realisation for $\mathcal{G}$ in the plane. We may assume, without loss of generality, that $\tau(s)$ is a reflection in the y-axis.

Suppose the symmetric 2-tree 1-extension which forms G' from G adjoins the vertices v and $s_{\theta'}(v)$. Then $d_{G'}(v) = 3$, and G is formed from G' by deleting v, $s_{\theta'}(v)$, and all edges incident to either of these vertices, before adding an edge e between the vertices in $N_{G'}(v)$ and another edge $s_{\theta'}(e)$ between the vertices in $N_{G'}(s_{\theta'}(v))$.

Suppose $d_{T_1'}(v) = 1$ and $d_{T_2'}(v) = 2$. For a pair of vertices $u, w \in V(G)$, write $u \sim^s w$ if either $u \in \{w, s_\theta(w)\}$, or, u is joined to either w or $s_\theta(w)$ by a sequence of parallel edges in $G - \{e, s_\theta(e)\}$. The construction of p' now follows the proof of Proposition 4.2 almost verbatim by replacing $\sim$ with $\sim^s$ and setting $p'(s_{\theta'}(v)) = \tau(s)(p'(v))$. The case where $d_{T_1'}(v) = 2$ and $d_{T_2'}(v) = 1$ can be proved by similar methods. $\square$

Theorem 5.10. Let $\mathcal{G} = (G; T_1, T_2; \theta) \in \mathcal{G}_2^{sym}$ be a symmetric 2-tree decomposition with $\mathcal{G} \neq \mathcal{K}_1$. Then $\mathcal{G}$ has a $\mathcal{C}_s$-realisation in the plane.

Proof. Use Propositions 5.8 and 5.9, and apply a similar argument to the proof of Theorem 4.3. $\square$

Theorem 5.10 shows that it is always possible to construct examples of isostatic $\mathcal{C}_s$-symmetric frameworks in the ℓ^∞ plane which induce prescribed symmetric monochrome spanning trees with no fixed edges. In the following, note that not all $(2,2)$-tight $\mathbb{Z}_2$-symmetric graphs admit a symmetric 2-tree decomposition.

Corollary 5.11. Let (G, θ) be a $\mathbb{Z}_2$-symmetric simple graph. If (G, θ) admits a symmetric 2-tree decomposition $\mathcal{G} = (G; T_1, T_2; \theta)$, with no fixed edges, then there exists a $\mathcal{C}_s$-realisation for $\mathcal{G}$ in the plane which is well-positioned and minimally rigid in $(\mathbb{R}^2, \|\cdot\|_\infty)$.

Proof. By Theorem 5.10, the symmetric 2-tree decomposition $\mathcal{G}$ has a $\mathcal{C}_s$-realisation in the plane. By Theorem 2.1, this realisation is minimally rigid in $(\mathbb{R}^2, \|\cdot\|_\infty)$. $\square$

6 Open problems

In Section 4, we showed that given any multi-graph with a partition of its edge set into two spanning trees, there exists a realisation of this 2-tree decomposition in the plane. It is not known whether this result extends to d-dimensions. Indeed, a solution here would settle a particular case of another open problem which is to determine whether every (d, d)-tight graph has a rigid placement in $(\mathbb{R}^d, \|\cdot\|_q)$ for $d \geqslant 3$ and $q \neq 2$.

Open Problem 5 (Rigidity for $\boldsymbol{\ell^q}$ norms.)**.** Let G be a simple graph which is an edge-disjoint union of d spanning trees $T_1, \ldots, T_d$, where $d \geqslant 3$.

(a) Does there exist a placement of G in $\mathbb{R}^d$ such that the induced monochrome subgraphs of G are precisely $T_1, \ldots, T_d$?

(b) Does there exist an isostatic placement of G in $(\mathbb{R}^d, \|\cdot\|_q)$ for all (or for some) $q \neq 2$?

A positive answer to (a) would imply a positive answer to (b) in the case $q = \infty$. If $d = 2$, then the answer to both questions is "yes".

The following example highlights the geometric difficulties which can arise in extending the constructive method presented here to higher dimensions. However, it may still be possible to solve the realisation problem for $d \geqslant 3$ by adapting these geometric arguments.

Example 6. Suppose $\mathcal{G} = (G; T_1, T_2, T_3)$ is a 3-tree decomposition with a realisation (G, p) in $\mathbb{R}^3$. Suppose x, y, z are vertices of G with $p(x) = (0, 0, 0)$, $p(y) = (-1, 3, -1)$ and $p(z) = (-1, 10, -3)$. Now suppose a 3-tree 0-extension is applied to $\mathcal{G}$ at the vertices x, y, z, which adds the vertex v and results in a 3-tree decomposition $\mathcal{G}' = (G'; T_1', T_2', T_3')$ with $xv \in T_1$, $yv \in T_2$ and $zv \in T_3$. For these trees to correspond to the monochromatic trees induced by a framework colouring for G', we must have

$$\begin{aligned}
\|p(v) - p(x)\|_\infty &= |p(v)_1 - p(x)_1| = |p(v)_1|, \\
\|p(v) - p(y)\|_\infty &= |p(v)_2 - p(y)_2| = |p(v)_2 - 3|, \text{ and} \\
\|p(v) - p(z)\|_\infty &= |p(v)_3 - p(z)_3| = |p(v)_3 + 3|.
\end{aligned}$$

However, there is no such point $p(v) \in \mathbb{R}^3$.

A related problem is that of constructing examples of redundantly rigid frameworks in $(\mathbb{R}^d, \|\cdot\|_q)$. Here a framework is *redundantly rigid* if it is rigid and every subframework obtained by the removal of a single edge is also rigid. Such frameworks have played a key role in the study of *global rigidity* for frameworks in Euclidean space (see for example [4]).

Open Problem 7 (Redundant rigidity for $\boldsymbol{\ell^q}$ norms.)**.** Let G be a simple graph which is an edge-disjoint union of d Hamilton cycles $H_1, \ldots, H_d$, where $d \geqslant 2$.

(a) Does there exist a placement of G in $\mathbb{R}^d$ such that the induced monochrome subgraphs of G are precisely $H_1, \ldots, H_d$?

(b) Does there exist a redundantly rigid placement of G in $(\mathbb{R}^d, \|\cdot\|_q)$ for all (or for some) $q \neq 2$?

A positive answer to (a) would imply a positive answer to (b) in the case $q = \infty$.

Similar realisation problems arise for other norms. For example, it is shown in [7] that rigidity for the cylinder norm on $\mathbb{R}^3$ is characterised by an induced framework colouring which decomposes the graph into an edge-disjoint union of a spanning tree and a spanning Laman graph. Again whether the existence of such a decomposition implies the existence of a geometric realisation is open.

Open Problem 8 (Rigidity for the cylinder norm.)**.** Let G be a simple graph which is an edge-disjoint union of two spanning subgraphs T and L where T is a tree and L is a Laman graph.

(a) Does there exist a placement of G in $(\mathbb{R}^3, \|\cdot\|_{cyl})$ such that the induced monochrome subgraphs of G are precisely T and L?

(b) Does there exist an isostatic placement of G in $(\mathbb{R}^3, \|\cdot\|_{cyl})$?

A positive answer to (a) would imply a positive answer to (b). The smallest graph in this class is $K_6 - e$, obtained by removing a single edge from the complete graph K_6, and this graph does admit an isostatic placement in $(\mathbb{R}^3, \|\cdot\|_{cyl})$ (see [7]).

Realisation problems of this type also arise in considering *forced symmetric rigidity* (see for example [5, 11, 13] for the Euclidean context). A characterisation is obtained in [10] for forced reflectional symmetry in the ℓ^∞ plane which is expressed in terms of framework colourings on the associated gain graphs. In this case, the gain graph is expressed as an edge-disjoint union of a spanning unbalanced map graph and a spanning tree.

Open Problem 9 (Forced symmetric rigidity for the $\boldsymbol{\ell^\infty}$ norm.)**.** Let G_0 be a gain graph for a $\mathbb{Z}_2$-symmetric graph which is expressible as an edge-disjoint union of two spanning subgraphs T and M, where T is a tree and M is an unbalanced map graph.

Does there exist a placement of the covering graph G in the plane, with reflectional symmetry, such that the induced monochrome subgraphs of G_0 are precisely T and M?

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