Pre-service Primary Teachers' Geometric Thinking: Is Pre-tertiary Mathematics Education Building Sufficiently Strong Foundations?

Abstract

Teacher knowledge is a critical focus of educational research in light of the potential impact of teacher knowledge on student learning. The dearth of research exploring entry-level pre-service teachers' geometric knowledge poses an onerous challenge for mathematics educators in initial teacher education (ITE) when designing experiences that develop pre-service teachers' geometric knowledge to support the task of teaching. This study examines the geometric thinking levels of entry-level Irish pre-service primary teachers (n=381). Participants' geometric thinking levels were determined through a multiple-choice geometry test (van Hiele Geometry Test (VHGT)) prior to commencing a Geometry course within their ITE programme. The findings reveal limited geometric thinking among half of the cohort and question the extent to which pre-tertiary experiences develop appropriate foundations to facilitate a smooth transition into ITE mathematics programmes. The study also examines the nature of misconceptions among those with limited geometric thinking and presents suggestions for appropriate ITE response.

Key words: geometric thinking, initial teacher education (ITE), pre-service primary teachers, van Hiele, teacher knowledge

Introduction

This paper examines pre-service primary teachers' geometric thinking. Geometric thinking is more than the ability to do geometry tasks. The term is used to refer to students' approaches to reasoning about shapes and other geometric ideas (van Hiele, 1999).

Geometry has the potential to develop important and transferable knowledge and skills including "...visualisation, critical thinking, intuition, perspective, problem solving, conjecturing, deductive reasoning, logical argument and proof" (Jones, 2002, p. 125). Consequently, geometry is addressed within both primary (elementary) and secondary (high school) mathematics curricula (Chinnappan & Lawson, 2005; Marchis, 2012; Yanik, 2011). Internationally, there is growing interest in geometry teaching and learning. This may be attributed to discontent with students' geometric understandings revealed by international comparative tests such as Trends in International Mathematics and Science Study (TIMSS) and Programme for International Student Assessment (PISA) (Close, 2013; Halat & Sahin, 2008; Steele, 2013). It is within this context that the importance of the teacher in affecting student experience and achievement has been repeatedly acknowledged (Conference Board of the Mathematical Sciences (CBMS,) 2001; Mewborn, 2001) leading to the consensus that "how students learn geometry greatly depends on the teachers..." (Unal, Jakubowski & Corey, 2009, p. 998). This has led to increased focus on research into the geometric knowledge of practicing and pre-service teachers (Halat, 2008; Steele, 2013).

In the last number of decades there has been increased interest in the knowledge required to teach mathematics effectively. Understandings of what constitutes mathematical knowledge required for teaching have evolved. There is agreement that a mathematics teacher requires deep knowledge of mathematics (Hill, 2010; Couto & Vale, 2014). It is now widely accepted that effective mathematics teaching requires mathematics knowledge for teaching (MKT) which has been described as 'knowledge, skills and habits of mind' entailed in the work of teaching (Ball et al., 2007). More specifically, this knowledge comprises of both *subject matter knowledge* (SMK) and *pedagogical content knowledge* (PCK) (c.f. Ball, Hill & Bass, 2005; Ball et al., 2007; Ball, Thames & Phelps, 2008; Hill, Schilling & Ball, 2004). SMK is also referred to as content knowledge and describes 'the amount and organisation of knowledge per se in the mind of teachers' (Shulman, 1986, p. 9). When referring to primary teachers' SMK, there is agreement that teachers' mathematical understandings should be substantially different from what any person working in other mathematics intensive professions (e.g., engineering) would possess (Hill et al., 2004; Ball et al., 2005). According to Ball et al. (2005), SMK can be further categorised into *common* and *specialised* content knowledge. While all educated people including mathematics teachers require *common content knowledge* (CCK) (an ability to do mathematics) as part of their job, it is considered that mathematics teachers also require *specialised content knowledge* (SCK) which is content knowledge for teaching; for example, awareness of multiple strategies to teach a concept (Ball et al., 2007). In addition to SMK, pedagogical content knowledge (PCK) is necessary to transform personal SMK into a form which makes it accessible to learners (for example, the most effective analogies, examples, and explanations) (Hill et al., 2004).

This paper focuses on pre-service primary teachers' geometric thinking. While frameworks exist, such as the van Hiele hierarchy of geometric thinking (van Hiele, 1999), which identify a learner's level of geometrical thinking, the authors acknowledge that these levels cannot be considered *de facto* indicators of an individuals' CCK. We do contend, however, that ones' level of geometric thinking shapes and impacts an individuals' potential to develop appropriate geometric SMK and in turn MKT.

Impetus for the study

This study was motivated by the authors' (mathematics teacher educators) experience of what they perceived to be primary pre-service teachers' under-developed geometric thinking. On further investigation of the relevant research, while there was agreement that primary teachers should possess appropriate geometric knowledge (SMK), there is limited research examining the geometric facility evident among primary teachers. However, there was consensus within the research that primary teachers, whether prospective or qualified, demonstrate deficient geometric thinking particularly in light of their role and potential influence within the classroom (Couto & Vale, 2014; Gomes, 2011; Jones, Mooney & Harries, 2002; Knight, 2006; Lin, Luo, Lo & Yang, 2011; Marchis, 2012; Mayberry, 1983; Swafford, Jones & Thornton, 1997; van de Sandt & Nieuwoudt 2005). However, in many of the samples examined in research, deficient geometric thinking was explained by limited opportunities to study geometry themselves in school (Ng, 2011; Watson, 2012). This is not the case, however, within the Irish context where all entrants to ITE have studied geometry as part of their mathematics syllabus for a minimum of 13 years during their pre-tertiary education. These students also have achieved high standards within their terminal secondary school examinations and achieved minimum entry requirements in mathematics (Central Applications Office (CAO), 2015; Hourigan & O' Donoghue, 2013, 2015; Hourigan et al., 2016; Hourigan and Leavy, 2017; Leavy et al., 2017). In light of these factors alongside ongoing anecdotal evidence, the authors sought to gain insights into the status of pre-service primary teachers' geometric thinking within the context of an ITE programme in Ireland. This study provides a unique contribution to the existing body of research; given the unique characteristics of the participants and the large sample size involved in the study.

Theoretical Perspective

Van Hiele Hierarchy of Geometric Thinking and its Philosophy

A widely accepted theoretical framework to explore the geometric learning process is the van Hiele hierarchy of geometric thinking (Unal et al., 2009; van Hiele, 1999; Wang & Kinzel, 2014). The Dutch educators Pierre and Dina van Hiele (van Hiele, 1986) proposed a model for geometric thinking that explains the development of human geometric thinking in response to difficulties students encountered (Watson, 2012). The framework suggests that learners must progress through a sequence of five discrete, qualitatively different and hierarchical levels of geometric thinking (Wang & Kinzel, 2014) (see Figure 1).

[INSERT FIGURE 1]

At the first level, *Level-0, the visual level* (also called Visualisation or Recognition), learners recognise figures judging them solely on overall appearance, for example, a child may say 'It is a square because it looks like a window'. They do not perceive geometric properties of figures (e.g., sides of equal length, right angles, and equal diagonal lengths). Although a learner at the next level, *Level-I, the descriptive level* (or Analysis), can identify a figure's properties (for example, number of sides, types of angle) and use these for the purpose of classification, there is no awareness of the relationships between these properties (e.g., do not perceive any relationship between rectangles and parallelograms). At the third level, *Level-II*, the *informal deductive level* (Ordering), properties are logically ordered. Learners at this level may deduce one property from another and make informal arguments to justify their response to conjectures (e.g., squares are special types of rhombi). At the next level (*Level-III*), *formal deduction* (Deduction or Formal Logic), theorems within an axiomatic system are established. Students can prove theorems deductively, construct proofs, and understand the role of axioms and definitions. At the uppermost level (*Level-IV*), the *meta-mathematical level* (Rigour), the learner is able to manipulate geometric statements such as axioms, definitions and theorems, and compare axiomatic systems. Understandings at this level compare to those of a mathematician and these learners possess understandings of non-Euclidean systems. van de Sandt and Nieuwoudt (2003) noted that the first three van Hiele levels identify thinking within the capacity of primary school learners, whereas the last two levels typically represent thinking required in high school or college level courses (Halat, 2008; Hoffer, 1988).

According to the van Hiele framework, each thinking level must be conquered sequentially and things learned in one level are objects to be studied at the next level; that is what is implicit at one level is explicit at the next level. Van Hiele (1999) proposed that progression from one level to another is more dependent on instruction and learning than age or biological maturation. This theory supports Piaget's tenet that "no education is better than giving it at the wrong time" (p. 310); in other words, the appropriateness of educational experiences is thought to potentially foster or alternatively impede development between levels (Couto & Vale, 2014; Hourigan & Leavy, 2015; van de Sandt & Nieuwoudt, 2005; van Hiele, 1999). In outlining the theory, the van Hieles present a description of each level and suggestions regarding ways to support learners to advance to the next level. This highlights the importance of the teacher in providing suitable geometrical experiences (Leavy and Hourigan, 2015). The acknowledged influence of the teacher in promoting geometrical thinking motivated the researchers in selecting the van Hiele hierarchy as a guiding framework, and assessment tool, in this study.

While initially the van Hiele theory was limited to two-dimensional shapes, research since the early 1980s has helped to confirm the validity of the theory (Halat, 2007; 2008; Mayberry, 1983, Usiskin, 1981) and extend the theory to other geometric domains. Criticism of the model suggests that it lacks depth and that little detail exists regarding the type of reasoning that exists at each level (Wang & Kinzel, 2014). The static view of geometry communicated by the discrete levels has also been disputed (Battista, 2007). Gutiérrez, Jaime and Fortuny (1991) instead theorize that learners may straddle or be in transition between levels and use different levels of thinking depending on the problem being presented. The existence of degrees of acquisition (e.g., low, intermediate, high acquisition) within each van Hiele level (van der Sandt & Nieuwoudt, 2003) has also been proposed. Despite these criticisms of the van Hiele theory, the theory is considered to correctly, although generally, depict the development of geometry thinking especially about shapes and in turn have "...a strong ring of validity" (Battista, 2007, p. 856). Other perspectives consider that learners who have not yet reached the first van Hiele level (Level-0: Visualisation) should be located within a pre-recognition level. This level acknowledges learners who are unable to recognise many common shapes from non-exemplars or who may be unable to distinguish between similar figures (e.g., unaware of the difference between a square and rectangle) (Dunphy, Dooley & Shiel, 2014; Halat & Sahin, 2008; Usiskin, 1982). Studies which include the pre-recognition level label this pre-Van Hiele level as Level-0 and the 5 van Hiele levels are labelled Level-I through to Level-V. This modified numbering of the van Hiele levels is used in the majority of relevant research. In light of this, Level-I through to Level-V is used to represent the 5 van Hiele levels in this study.

Pre-service Teachers' Geometric Thinking

Teacher educators strive to develop MKT. As school mathematics (pre-tertiary) experiences focus on the development of learner knowledge, it is understandable that pre-service teachers entering ITE would not demonstrate the types of knowledge expected of a teacher (i.e., SCK) (CBMS, 2001; Mewborn, 2001; Ng, 2011). However, it is concerning that many international studies have also reported that pre-service primary teachers' personal mathematics subject matter knowledge is limited (Ball, 1990; CBMS, 2001; Hourigan & O' Donoghue, 2013, 2015; Leavy & O' Loughlin, 2006; Leavy, 2015; Ng, 2011). While there is a wealth of studies examining both qualified and pre-service primary teachers' mathematics subject matter knowledge, there is relatively little research focusing specifically on the level of geometric subject matter knowledge primary teachers possess on entry to, during or on exit from ITE (Gomes, 2011; van de Sandt & Nieuwoudt, 2005). In fact, Steele (2013) reports that research into teacher knowledge related to geometry have been almost non-existent.

However, our review of the existing research reveals dissatisfaction internationally regarding the geometric thinking demonstrated by pre-service primary teachers (Couto & Vale, 2014; Gomes, 2011; Jones, Mooney & Harries, 2002; Knight, 2006; Lin, Luo, Lo & Yang, 2011; Marchis, 2012; Mayberry, 1983; Swafford et al., 1997; van de Sandt & Nieuwoudt 2005).

While many of the studies that focus on pre-service primary teachers' geometric thinking developed qualitative instruments deemed fit-for-purpose (e.g., Couto & Vale, 2014; Wang & Kinzel, 2014; Yanik, 2011), a large number of studies used the van Hiele Geometry Test (VHGT) (Usiskin, 1982) (e.g., Halat & Sahin, 2008; Karakus & Perker, 2015; Mayberry, 1983; Swafford et al., 1997; Unal et al., 2009; van de Sandt & Nieuwoudt, 2005). The majority of these studies used the VHGT test outcomes to assign pre-service teachers to one level (i.e. the highest van Hiele level they demonstrated). For example, Mayberry (1983) found that 13% of primary pre-service teachers were below Level-I, 20% at Level-I (Visual level), 19% at Level-II (Descriptive level), 24% at Level-III (Informal deduction level), and 25% at Level-IV (Formal deduction). Similarly, Swafford et al. (1997) reported that 79% of pre-service teachers were functioning at or below van Hiele Level-III (Informal Deduction or less). Knight (2006) also reported that pre-service primary teachers' thinking levels were below Level-III. Halat and Sahin (2008) reported that none of the pre-service primary teachers in their study demonstrated van Hiele Level-IV or Level-V geometric thinking. In fact 74% of these pre-service primary teachers demonstrated thinking below van Hiele Level-III thinking (Informal Deduction), thus mirroring the findings of Swafford et al. (1997). Interestingly, cross-cultural differences may exist as reported by Lin et al. (2011) who revealed substantial differences between the van Hiele levels evident among Taiwanese and U.S. pre-service primary teachers, with the former outperforming the latter.

Overall, the research concurs that pre-service primary teachers demonstrate low levels of geometric thinking. In terms of those utilising the VHGT, this was particularly evident from the fact that a substantial group of pre-service primary teachers were unable to demonstrate van Hiele Level-III geometric thinking (Informal Deduction) (Bulut & Bulut, 2011; Halat & Sahin, 2008; Ng, 2011). The findings are particularly disconcerting in light of the contention by Petrou (2007) that teachers are unsuccessful in promoting mathematical thinking outside the limits of their own facility. Accordingly, Unal et al. (2009) question whether teachers who have not reached van Hiele Level-IV thinking lack the depth of geometric knowledge needed to teach effectively.

Factors contributing to weak geometric thinking. In terms of the causes, research suggests that inadequate mathematics subject matter knowledge among entrants to ITE are consequences of systemic rather than personal failings (Ball et al., 2005; CBMS, 2001). In some countries, students may not have studied geometry at secondary level (Ng, 2011). For example in the US, Watson (2012) reports that due to areas such as algebra being prioritised after elementary systems, geometry is not usually revisited until 10th grade resulting in students entering high school with elementary level geometric thinking. Another source of discontent is the reality of varying geometry curricula across school districts and jurisdictions (Watson, 2012). This is not the case within the Irish context. All pre-service teachers have studied geometry throughout their primary and secondary school education – for a minimum of 13 years. The final two years of secondary education; the *Leaving Certificate course;* are examined by means of the Leaving Certificate examination which can be studied at *Foundation (F) level, Ordinary (O) level* or *Higher (H) level*. At each of these levels, grades assigned range from A1to D3 (a student may achieve A1, A2, B1, B2, B3, C1, C2, C3, D1, D2, D3, F). However, there is consensus that the examination-led system in place until very recently, which promoted a focus on memorisation of rules and procedures, has short-changed Irish students entering college-level mathematics courses. While all entrants to primary ITE have met the minimum mathematics entry requirements (i.e. a D3 grade on Ordinary (O) or Higher (H) level mathematics in the terminal examination (Leaving Certificate)), students across the board demonstrate substandard mathematical knowledge (Hourigan & O' Donoghue, 2007, 2013, 2015; Leavy & O' Loughlin, 2006; Leavy, 2015; National Council for Curriculum and Assessment (NCCA), 2006). The perceived inadequacy is reflected in the proposal from the national Teaching Council, the body responsible for accrediting teachers, that the minimum mathematics entry requirement should be raised (Teaching Council 2011). Parallel efforts to address mathematics subject matter knowledge deficiencies focus on the reform of pre-tertiary curricula (NCCA 2006). A new second-level (high school) curriculum called Project Maths, introduced incrementally in phases at a national level from September 2010, is designed to reform second level mathematics education in a bid to promote conceptual understanding and problem solving within realistic contexts as well as support a smooth transition from primary through second level and onto college level mathematics education (NCCA, 2006; 2012). Although at an early stage of implementation, initial reports are tentatively positive (Jeffes et al., 2013). Once the entire Project Maths syllabus (both Junior and Senior Cycle) is rolled out, established and implemented as intended, it is possible that future prospective teachers enter ITE programmes demonstrating more compatible learner knowledge than previously demonstrated in studies. However, there have been no large scale evaluation studies examining the fidelity of implementation of the geometry components of reform curricula. Indeed, there is international research which reveals that innovations (such as reform curricula) are often adapted to suit local needs and these threats to the fidelity of implementation occur frequently (Rogers, 2003). Thus we cannot assume adherence to the quality of delivery *Project*

Mathematics until presented with evidence from evaluation studies or data indicating differences in student outcomes as a result of engagement with reform curricula.

Implications for initial teacher education. When students' only exposure to mathematics is limited to didactic teaching approaches focusing primarily on computation and procedures, what results is limited and shallow mathematical knowledge (Ball, 1990; CBMS, 2001; Mewborn, 2001; NCCA, 2006; van de Sandt & Nieuwoudt, 2005). Limited mathematics SMK among teachers is associated with less competent mathematics teaching where they rely on the textbook for content, prioritise passive seat work over interactive methodologies and portray mathematics as a set of static isolated skills to be learned (CBMS, 2001; Gomes, 2011; Mewborn, 2001; Steele, 2001). Therefore it is a critical undertaking that ITE mathematics educators gain insights into the nature of pre-service teachers existing learner knowledge on entry to geometry courses. This will, in turn, inform the development and implementation of mathematics education programmes to best meet the needs of participating pre-service teachers.

Context of this Study

This study is situated in a College of Education (ITE institution) in Ireland where the researchers are mathematics teacher educators. This research study is part of a large-scale study which examines the characteristics of pre-service primary teachers (Leavy, Hourigan, & Carroll, 2017) and the impact of participation in a mathematics education programme (consisting of a series of 5 compulsory modules) on these characteristics (Hourigan, Leavy, & Carroll, 2016; Hourigan & Leavy, 2017). As part of informal observations and interactions during mathematics education modules over many years, the researchers perceived pre-service teachers' geometric knowledge to be disjointed and shallow. For example, many pre-service teachers only considered prototypical shapes when referring to shapes with 5 or more sides (e.g., only regular pentagons, hexagons). Further insight was gained from a previous investigation into the nature of Irish pre-service primary teachers' mathematics knowledge. While this study did not focus specifically on geometry, the findings in relation to a mathematics test item on geometry provided additional support for the researchers' beliefs. Almost ¾ of the pre-service primary teachers in the study either disagreed or could not justify their agreement with the statement 'A square is a special type of rectangle. True or false (T/F). Explain your answer' (Hourigan & O' Donoghue, 2013, 2015). Responses included *'A square is a square. Definition of square all sides are equal- doesn't hold true for a rectangle'* and '*True because both have two pairs of parallel lines'.*

In this paper, researchers explore the level of geometric thinking pre-service teachers bring with them from pre-tertiary mathematics education into ITE. The ITE mathematics courses provide opportunities for pre-service teachers to develop geometric SMK alongside PCK; however, in order for these experiences to be optimally beneficial, instruction must be needs-led building on participants' prior knowledge (Gomes, 2011; van de Sandt & Nieuwoudt, 2005). Therefore, this study responds to Halat's (2008) recommendation that mathematics teacher educators ought to examine pre-service teachers' geometric thinking in order to develop a course, which will promote optimum growth in this regard.

Methodology

Participants

Participants were pre-service primary teachers enrolled in a 4-year ITE degree programme in a College of Education in Ireland; the intake to this ITE programme constitutes 50% of the yearly intake of undergraduate pre-service primary teachers in Ireland. In Ireland, the college applications procedure is centralised. Therefore, instead of applying to individual institutions, the Central Applications Office (CAO) is the organisation responsible for overseeing most undergraduate applications, where points are allocated to each grade achieved for the student's best six subjects in the Leaving Certificate examination. As primary teaching is held in high esteem in Ireland, despite a large number of places on the programme (approximately 460 annually in the ITE institution in this study) the high demand for places to primary level ITE programmes results in entrants ranking in the top 15% of all Irish college entry students (Central Applications Office (CAO), 2015).

All participants had studied geometry (alongside other areas such as number, algebra, and statistics) throughout their 13 years of pre-tertiary education (see Table 1). While they experienced established geometry curricula at both primary and junior cycle post-primary education, they experienced the first phase of the reform Project Maths geometry syllabus (final 2 years of secondary school).

[INSERT TABLE 1]

Participants were in their second year of the degree programme and had completed 3 of the 5 compulsory mathematics education modules. This research examines the level of geometric thinking demonstrated by these pre-service teachers prior to enrolling in their fourth mathematics education module, which focused specifically on Geometry.

Participation was voluntary and complete anonymity was guaranteed to all participants through the use of a unique project ID. Ethical approval was granted by the College Ethics Committee. All 445 of the 2nd year cohort of students were invited to participate in the research. A total of 381 pre-service teachers volunteered, 31% were male (n=91) and 69% female (n=290). This represents broadly the gender distribution within the student cohort.

Research Hypothesis and Question

In this study, the van Hiele theory provides a lens to explore the geometric thinking that pre-service primary teachers bring with them from pre-tertiary mathematics education on entry to an ITE geometry education module. The following research hypothesis (H) and research questions (Q) guide the study:

H1: Pre-service teachers demonstrate low van Hiele levels of geometric thinking prior to commencing an ITE geometry education course

Q1: What are the geometric thinking levels of pre-service primary teachers prior to entering the ITE geometry education course?

Q2: Are mathematics grades in the secondary school terminal examination an indicator of geometric thinking (as indicated by the VHGT)?

Q3: Are there any patterns in responses on the VHGT which reveal weaknesses in geometric thinking with respect to specific geometric concepts?

Research Instrument

The researchers administered the van Hiele Geometry Test (VHGT) (Usiskin, 1982). The VHGT is a scientific test which was developed within Usiskin's (1982) study and is designed to determine the van Hiele level at which the student is working. It has proved an extremely popular instrument among all levels of mathematics education and has been used within contexts far exceeding its intended purpose (c.f. Usiskin, 1982; Usiskin & Senk, 1990). The test consists of 25 multiple-choice geometry questions, which are organised into blocks of 5 questions that have been created using behaviours from van Hieles' writings. The blocks are arranged sequentially and correspond to the respective van Hiele levels from the lower to higher levels. Therefore, while questions 1-5 correspond directly to the characteristics of van Hiele Level-I thinking, questions 6-10 match with the descriptions of van Hiele Level-II thinking, questions 11-15 with van Hiele Level-III thinking and so on for all five levels (Usiskin, 1982). A brief description of the nature of questions in the respective levels can be found in Karakus and Perker (2015, p. 344). In this study, participating pre-service teachers' VHGT attainment of levels I-IV was investigated (20 items). Participants' attainment of Level V reasoning was not investigated, as such geometric reasoning levels were not considered essential for primary teachers. Another consideration in this decision was concern regarding the ability of VGHT Level V items to test participants' geometric thinking at this level (c.f. Usiskin, 1982, p. 32). Therefore the modified case (based on the first four distinct levels) was used. In addition, the term pre-recognition (Level 0) was used to describe participants who did not reach van Hiele level I (Usiskin, 1982).

Alternative instruments such as the Content Knowledge for Teaching Mathematics (CKTM) measures developed by the Learning Mathematics for Teaching (LMT) Project at the University of Michigan (Hill et. al., 2004) were examined for use in this study. Given that items focus on both CCK and SCK, and are framed within the teaching-specific context, the CKTM was not considered suitable for our participants as they were pre-service teachers who had not yet received instruction on pedagogies relating to the teaching of geometry. Sorting tasks and clinical interviews were also considered (Battista, 2007; Wang & Kinzel, 2014) on the basis that they would provide a more detailed picture of the participants' geometric thinking, such approaches were discounted on the basis that it was not feasible to implement these time-consuming approaches to such a large cohort of students. The paper and pencil instrument selected (VGHT) was more practical in order to efficiently gain insights into the geometric thinking levels of those entering the ITE geometry education module.

Instrument administration and analysis. Efforts were made to minimise threats to internal and external validity. The validity and reliability of the data collection instrument used has been justified elsewhere (c.f. Usiskin, 1982; Usiskin & Senk, 1990). In order to combat the instrument threat, we prepared instructions for use of the VHGT. Participants were given 40 minutes to complete the test. Data collector threat was controlled by ensuring that the data collection instruments were administered by the same person. Student profile information (e.g., pre-tertiary terminal mathematics grade) provided the researchers with predictive validity evidence for the future (i.e., the extent that these factors predict participants' performance on the VHGT) (Nitko, 2001).

When scoring responses the researchers adopted the Level I-IV numbering for the van Hiele levels (instead of 0-III). This facilitated assigning Level-0 (pre-recognition) to participants who did not function at Level-I. Each participant received a score guided by Usiskin's (1982) grading system: one point for every correct answer, zero points for every incorrect or nonresponse. In this study, the "3 of 5 correct criterion" was selected for success in attaining any given van Hiele level (thus addressing Type II error). This meant that if a participant answered at least 3 (out of 5) items correct in a given block, he/she was considered to have mastered that level (Usiskin, 1982, p. 23). The stricter criterion (4 of 5 correct criterion) was also considered given that it ensured a higher mastery level and minimised the risk of participants guessing (thus addressing Type I error) (Usiskin, 1982). However given that the participants had not studied geometry formally since the end of their secondary education (over 18 months prior), the researchers believed the criterion selected (3 of 5 correct) to be more appropriate. Therefore the study is tolerant to the 3 out of 5 criterion in order to avoid the situation wherein a participant has capability to master a higher van Hiele level being misplaced into a lower van Hiele level. In assigning a van Hiele level, participants were initially assigned a weighted sum score based on correctly answering the allotted number of questions (minimum 3 out of 5) in each block. They received 1 point for meeting criterion on items 1-5 (Level-I), 2 points for meeting criterion on items 6-10 (Level-II), 4 points for meeting criterion on items 11-15 (Level-III) and 8 points for meeting criterion on items 16-20 (Level-IV).

However, possessing a higher weighted total score does not necessarily mean that students demonstrated a higher van Hiele level. This is because based on the van Hiele theory students must go through the levels sequentially without leaving any out. The weighted score total was used to assign each participant a forced van Hiele level. The forced method was selected given the reality that many participants who achieved level I and III (weighted sum= 5) on the VHGT would be categorised as 'no fit' if the classical or modified method were used. Usiskin (1982, p. 35) justified this strategy, stating "Forcing a van Hiele level is tantamount to assuming that the theory does hold and that those who do not fit would have fit if there had been more items or better items to minimise random error classifications". A forced van Hiele Level is assigned Level 'n' if:

1. The student meets criterion (in this study '3 of 5') at level 'n', all levels below 'n' but not at level 'n+1' yet meets another higher level e.g., meets criterion for Levels I, II, IV (weighted score =11), assigned forced van Hiele Level-II
2. The student meets the criterion at levels (in this study '3 of 5') 'n' and 'n-1' but perhaps does not at one of the levels 'n-2' or 'n-3' e.g., Achieves criterion for Levels I, III, IV (weighted sum= 13), assigned forced van Hiele Level-IV.

The utilisation of the forced method to analyse the data within this study resulted in every participant being assigned a van Hiele level (Usiskin, 1982). The forced van Hiele table (c.f. Abdullah & Zakaria, 2013, p. 4438) was used as a reference to determine participants' level of geometric thinking.

Once participants were assigned a van Hiele level, all relevant data were input and screened for errors. Descriptive statistics such as frequencies, measures of central tendency (mean, mode), and variability (standard deviation) were generated. Inferential analysis was also completed to determine the strength of relationships between factors (cross tabulations). The selection of the type of analysis tool was determined by the type of data set and the normality of the distributions.

Discussion of Findings

The findings are structured according to the research hypothesis and questions. To develop a profile of the geometric thinking abilities of the 381 participants, responses were categorised within a forced van Hiele level (Usiskin 1982), in a bid to address the hypothesis; *Pre-service teachers demonstrate low van Hiele levels of geometric thinking prior to commencing an ITE geometry education course* and answer the research question *What are the geometric thinking levels of pre-service primary teachers prior to entering the ITE geometry education course?* Table 2 describes the proportion of participants within each van Hiele level.

[INSERT Table 2 HERE]

Given that the first three van Hiele levels are considered within the thinking capacity of primary school learners, it is assumed that primary teachers require a minimum of Level-III thinking (i.e. Level-III or higher) (Halat, 2008; Petrou, 2007; van de Sandt & Nieuwoudt, 2003). At the time of test administration, just over two fifths (45.9%, n= 175) of participants demonstrated such geometric thinking (see Table 2). While a low proportion, it is important to remain cognisant that these students had not studied geometry for at least 18 months prior to completing the VHGT. Furthermore, participants may not have been invested in completing the VHGT measure or in persevering while completing the instrument as their performance was not an integral component of their coursework and had no bearing on their grade for the geometry course. However, these factors are somewhat balanced by the use of the modified test (Level-V items omitted) as well as the 3 of 5 correct criterion.

Examination of the response patterns at low VHGT levels addressed the research question *'Are there any patterns in responses on the VHGT which reveal weaknesses in geometric thinking with respect to specific geometric concepts?* Four percent of participants (n=16) did not demonstrate sufficient geometric thinking to be categorised in the first van Hiele level and were categorised as Level-0 (Pre-recognition). It is only on analysing the nature of the 5 VHGT items used to test the presence of Van Hiele Level-I (Visualisation) thinking that the limitations of such participants' geometric thinking becomes apparent. Each of these items presents a series of figures and asks 'Which of these are …?' The target 2-dimensional shapes were triangles (item 2), squares (items 1, 4), rectangles (item 3), and parallelograms (item 5). A participant who failed to respond correctly on 3 of these 5 items is demonstrating fundamental misconceptions in relation to their geometric thinking. On analysis, common errors among this subgroup of respondents included identifying a given shape as belonging to a class of shapes due to its resemblance to the category of shape in question (e.g., categorising the shape in Figure 2a as a triangle). This indicates a lack of attention to the properties of shapes or *partitive classification* (Clements & Sarama, 2007). Other common errors involved precluding particular shapes from a category of shapes due to orientation (e.g., square, see Figure 2b) or not considering shapes outside of the prototype (e.g., parallelogram, see Figure 2c) (Fox, 2000; Hourigan & Leavy, 2015). The authors believed that the most likely reason for this level of difficulty was an over exposure to prototypical representations of shapes during pre-tertiary primary instruction.

[insert Figure 2 here]

One would not expect such limited knowledge from participants who had completed a minimum of 13 years geometry instruction. There is research to suggest that these responses are potentially a product of the nature of instruction within pre-tertiary education which is marked by an overreliance on textbooks which focus on prototypical shapes, an over-emphasis on shape recognition combined with limited exploration of the relationships between shapes (Hourigan & Leavy, 2015; Hourigan & O' Donoghue, 2013).

A further fifth (22.8%, n=87) of participants were identified as functioning at Level-I (Visualisation). Geometric thinking at this level is marked by the tendency to rely on perceptual input to support recognition rather than attending to the particular characteristics of shapes; a level of geometric thinking frequently associated with kindergarten students (Sarama & Clements, 2009). Thus those participants who did not reach Level-II did not demonstrate ability to identify and use properties of figures to categorise.

Achievement of Level-II thinking suggests an ability to identify critical attributes, which characterise each class of shapes. In this study, the findings suggest that almost three quarters of participants demonstrate this facility (see Table 2). Those who did not achieve Level-III in the VHGT did not demonstrate a consistent ability to logically order the relationships and connections between properties for the purposes of class inclusion. Students had particular difficulty with items 14 and 15, both of which examined the relationship between the square and rectangle. Examination of the distribution of item level responses suggests that participants' pre-occupation with the differences in physical appearance and non-salient attributes (e.g., the need for the rectangle to have 2 long sides and 2 short sides of the proto-type) overshadowed relationships between the shapes. This suggests a lack of conceptual understanding and a dependence on procedural knowledge and isolated facts. The findings collectively suggest that many participants' geometric thinking is underdeveloped.

As part of the consent form, which accompanied the VHGT, participants were invited to share information regarding achievement in their secondary school (pre-tertiary) mathematics terminal examination (Leaving Certificate). While geometry is one component of this examination; other components include algebra, statistics and probability, functions, number, and trigonometry. While students entering primary ITE courses in Ireland must achieve minimum entry requirements in mathematics (i.e. D3 at O/H level Leaving Certificate), the vast majority of students in this study far exceeded the cut-off point (see Table 3). Almost a third (31%, n=114) of participants achieved an A or B grade at Higher level and only 5% achieved a C grade or lower at Ordinary level mathematics.

[insert Table 3 here]

These data were used alongside the VHGT levels to address the question *Are mathematics grades in the secondary school terminal examination an indicator of geometric thinking (as indicated by the VHGT)?* A cross-tabulation of the Leaving certificate grades and the van Hiele levels was analysed. For this purpose Leaving Certificate grades were banded e.g., HA1 and HA2 were categorised together as HA etc. The existence of emerging trends and patterns were explored. Preliminary analysis suggested that, in general, higher proportions of those with higher Leaving Certificate grades were situated within higher van Hiele levels and vice versa. For example, while approximately 70% of HA (n= 7) and HB (n= 72) students achieved either Level-III or Level-IV in the VHGT, this level of geometric thinking was evident among less than half of HC (45.7%, n=27), HD (42.9%, n=3), and OA (43.3%, n=21) participants respectively. At the other extreme, the pattern persists where higher proportions of those with lower Leaving Certificate grades were situated within low van Hiele Levels (Level-0 or Level-I in the VHGT) than their peers with higher Leaving Certificate grades. For example while 57.9% of OC students (n=11) achieved Level-0 or Level-I in the VHGT, the percentage for OB, OA, HC, HB were 34.8%, 36.7%, 17%, and

8.7% respectively. The chi-square analyses revealed that assignment to VHGT van Hiele levels did significantly differ by Leaving certificate mathematics grades (χ^2 (28, N = 367) = 69.187, p = .000). However, while the Leaving Certificate performance appears to be related to performance on the VHGT, some exceptions existed. In these cases, some participants who had achieved high Leaving Certificate mathematics grades demonstrated limited geometrical thinking. For example, 20% of HA students demonstrated Level-I thinking in the VHGT.

Conclusions

Analysis of the findings presents a relatively stark picture of the level of geometric thinking that prospective primary teachers possess (see Table 2) in light of the extent of their pre-tertiary geometric experiences and their relatively high scores in national assessments of mathematics (see Table 3). The findings justify the researchers' initial concerns that motivated this study in that a relatively high proportion of pre-service teachers are entering ITE with lower than expected levels of geometric thinking. These low levels of geometric thinking support the outcomes of research carried out by Mayberry (1983) and Swafford et al. (1997).

There was significant variation in the levels of geometric thinking demonstrated across participants. This variation may reflect the gaps which exist between the intended, implemented, and attained curriculum in secondary school and may raise questions about the fidelity of implementation of the reform mathematics curriculum across different secondary schools. While the findings suggest low levels of geometric thinking among over half of participants (as reflected by VHGT levels) (see Table 2), the terminal examination (Leaving Certificate) mathematics grades was found to be related to the Van Hiele level achieved. However, there were a substantial number of participants, across all Leaving Certificate grades, who demonstrated extremely low van Hiele levels of geometric thinking. This finding questions the reliability of the Leaving Certificate mathematics examination as a valid means of ensuring that prospective primary teachers possess adequate geometric learner knowledge on entry to ITE. The researchers believe that poor performance in the VHGT among those who have achieved grades far exceeding the minimum requirements for entry to the degree programme is a further cause for concern.

While poor performance of pre-service teachers on the VHGT is less than satisfactory, and perhaps worse than anticipated, it is not unexpected given the nature of pre-tertiary mathematics experiences. The primary source of this problem may arise from the emphasis at post-primary level on procedural knowledge (reproduction), text-based and exam-oriented teaching (Hourigan & O' Donoghue, 2007, 2013; NCCA, 2006). Therefore, while participants may have learned to manipulate axioms and theorems, they may lack knowledge and understanding of the basics in terms of connections and relationships. The item level findings suggest that over-presentation of prototypical shapes promotes a focus on dominant visual differences between classes of shapes which hinders students openness to inclusion (e.g., parallelograms). The findings imply that participants need further opportunities to develop their understandings of the properties of classes of shapes (Level-II) and to deduct a property on the basis of another (Level-III). This may be achieved by providing opportunities for experiences in informal geometry prior to introducing more formal geometry. Into the future, the recent reform of mathematics education at post-primary level, if implemented appropriately may result in a smoother transition for future pre-service primary teachers entering ITE.

The outcomes of this study reinforce the importance of designing quality ITE courses in order for fragmented geometric knowledge to be addressed (Couto & Vale, 2014). Pre-service teachers require ongoing opportunities to develop their geometric thinking in order to support them in developing the requisite SMK and in turn MKT to equip them to adequately teach school level geometry. This is particularly the case for those participants in our study who performed at Level-0 (4.2%) or Level-I (22.8%) (see Table 2). In order to teach mathematics in primary school, it is considered necessary for these pre-service teachers to possess van Hiele Level-III or above (Halat & Sahin, 2008). The findings therefore provide important information for mathematics teacher educators in order to develop a needs-led geometry education course with appropriate learning opportunities for these pre-service teachers (Yanik, 2011). While not explored in this study, it is to be expected that participants' geometric thinking and understandings will improve following completion of a geometry course. A study of the effect of a geometry intervention (Swafford et al., 1997) found that 72% of teachers increased by at least one van Hiele level and 50% of participants increased by more than 2 levels. In addition to providing experiences in geometric content, we support Jones et al.'s (2002) contention that how students come to know mathematics is important also. Geometry courses for pre-service primary teachers must provide opportunities to develop their geometric thinking. We suggest that this can be best achieved by working through the 5 phase instructional strategy recommended by van Hiele (1999) which involve Inquiry, Direct Orientation, Elicitation, Free Orientation, and Integration. The example in Appendix 1 provides insights into considerations when structuring an instructional unit (in this case, focusing on the concept of quadrilaterals) and using the van Hiele (1999) 5 phase instructional strategy as a guiding framework. Elements of this framework help address the revealed shortcomings in participants' geometric thinking by providing the structure to support them in challenging their focus on non-salient physical attributes of shapes, exploring connections and relationships between properties of shapes, and gaining insights into class inclusion.

It is important to acknowledge the limitations of this study. Research suggests that students may exhibit different van Hiele levels for different concepts (Burger &

Shaughnessy, 1986; Mayberry, 1983). Due to the fact that the VHGT focuses on two dimensional shapes only, the results of the study cannot be generalised to all geometric concepts. Thus, no claim is made that the test is an absolute measure of participants' van Hiele level for all geometric concepts. Its purpose is simply to act as a self-audit for students themselves in addition to providing feedback to inform the development and revision of the geometry education module.

Acknowledging limitations, this research study has added to national and international research regarding the geometric thinking that primary pre-service teachers bring with them to ITE. While acknowledging existing international research, it is pertinent that each country seeks to evaluate and address the specific needs of pre-service teachers within that unique national context. The study also contributes to the field by providing insights into the nature of pre-service teachers' limited thinking as well as presenting means of addressing these within ITE. This study acts as base-line study, which provides a benchmark for ongoing study. A thorough analysis of items would provide insights into specific geometric misconceptions and in turn inform the development of a course optimally compatible with the needs of pre-service teachers. Furthermore, the findings presented in this paper could potentially provide base-line data upon which to ascertain the impact of an ITE geometry course on the development of geometric thinking through comparison of pre- and post-VHGT scores.

Appendix 1: Example of Geometry Session to promote pre-service teachers' geometric thinking

Instructional Phases	**Activities within ITE session focusing on Quadrilaterals**
Inquiry	Introduction: Students are asked to locate the shape from under their chair and hold it up if it is a quadrilateral. Students are shown a 4 sided shape, where one side is curved, and are asked if it is a quadrilateral. The ensuing discussion clarifies that a quadrilateral is a closed 2-D shape with 4 straight sides. Students are asked to re-consider their selection and hold up their shape if it is a quadrilateral. Free Sort: Students are presented with a set of quadrilaterals and work in pairs to classify the shapes using 3 different rules. The teacher circulates and interacts with students, guessing their rule. Students share their rules (or alternatively the set of shapes that meet the rule). The class may be asked to implement a given rule. Rules may focus on properties such as regularity, convexity, sides, lines, angles, symmetry.

Direct Orientation	Parallelogram Sort: A definition for a parallelogram: 'A quadrilateral with 2 pairs of parallel lines' is introduced. Students are asked to implement the rule on their set of quadrilaterals. Students then discuss the shapes that meet the parallelogram criteria. They are asked to share their observations- which shapes they expected and are surprised by (Square, Rectangle, Rhombus). Each of the shapes is discussed in turn addressing why they are part of the parallelogram family and what additional features they have e.g., Rhombus is a special type of Parallelogram where all sides are equal. Trapezoid Dilemma: Students examine the shapes outside the rule ring with respect to parallel lines. Shapes are subdivided into those with no parallel lines and 1 pair of parallel lines. Shapes with 1 pair of parallel lines are examined. The students are informed that shapes with 1 pair of parallel lines are called Trapezoids. An alternative definition of Trapezoid (At least 1 pair of parallel lines) is presented and students are asked to re-sort the quadrilaterals using this definition. The relationship between parallelograms and trapezoids is discussed (Parallelograms are part of the Trapezoid family). Rectangle Realities: The definition for a rectangle is revisited: 'A parallelogram with all angles equal (right angles)'. Again students are asked to classify the quadrilaterals placing the rectangles in the rule ring. Students are asked to discuss the findings – what shapes did they expect? What other shapes are present? What was their understanding in school? Minimum defining characteristics for a rectangle are considered- focusing on other properties which could be used e.g., 'Do we need to say opposite sides equal?'
Elicitation	Conjectures: Students work in pairs to respond to conjectures (and their converse) in order to share what they have learned e.g., 'True or false- A square is a rectangle- justify'; If a square is a rectangle is a rectangle a square? Students share their response.
Free Orientation	Shape Riddles: Students respond to riddles by putting the relevant shape into the rule ring (e.g., I have at least one pair of parallel lines). Each line of the riddle is introduced incrementally (e.g., I have acute angles). Students are asked to create their own riddle.

Integration	Reflective Journal: Students are encouraged to reflect on their learning- what were their understandings of quadrilaterals prior to engaging in the session? What were their experiences of quadrilaterals? What are their new understandings generated as a result of engagement in the activities?

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